MATHEMATICS

A Textbook for Class XI

Part III

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Chapters 1-12 have been covered in Parts I and II.

CHAPTER 13

Binomial Theorem

13.1 The Binomial Theorem

In this chapter, we shall prove and study an important theorem. Its statement involves the combinatorial numbers $C(n, r)$ introduced in Chapter 12. Its proof is by mathematical induction which we studied in Chapter 3.

This theorem gives a formula for the expansion of $(a+b)^n$ where n is any positive integer. It is called the binomial theorem, because as you already know, the expression $(a+b)$ is a binomial. This theorem gives a formula for the powers of binomial expression. We first recall the following known formulae:

$$\begin{aligned}
(a+b)^{\circ} &= 1 \\
(a+b)^1 &= a+b \\
(a+b)^2 &= a^2+2ab+b^2 \\
(a+b)^3 &= a^3+3a^2b+3ab^2+b^3.
\end{aligned}$$

By actually multiplying, we find

$$\begin{aligned}
(a+b)^4 &= a^4+4a^3b+6a^2b^2+4ab^3+b^4 \\
(a+b)^5 &= a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5.
\end{aligned}$$

The binomial theorem gives a general formula for $(a+b)^n$ for which all these become particular cases. By looking at the above particular cases, we easily guess that the general formula would be of the form:

$$(a+b)^n = a^n + c_1a^{n-1}b + c_2a^{n-2}b^2 + \ldots + c_{n-1}ab^{n-1} + b^n.$$

The description will be complete, had we known the coefficients $c_1, c_2, \ldots, c_{n-1}$. For this purpose, we make a closer observation of the coefficients of the above known expansions.

We list them in the form

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

We observe that these coefficients are the same as:

$$\begin{array}{ccccccccccc}
 & & & & & C(0,0) & & & & & \\
 & & & & C(1,0) & & C(1,1) & & & & \\
 & & & C(2,0) & & C(2,1) & & C(2,2) & & & \\
 & & C(3,0) & & C(3,1) & & C(3,2) & & C(3,3) & & \\
 & C(4,0) & & C(4,1) & & C(4,2) & & C(4,3) & & C(4,4) & \\
C(5,0) & & C(5,1) & & C(5,2) & & C(5,3) & & C(5,4) & & C(5,5)
\end{array}$$

Now we guess the statement of the *Binomial Theorem*:

The Binomial Theorem: For any positive integer n,

$(a+b)^n = C(n,0)a^n + C(n,1)a^{n-1}b + C(n,2)a^{n-2}b^2 + \ldots + C(n,n-1)ab^{n-1} + C(n,n)b^n$,

where $C(n,r) = \frac{n!}{r!(n-r)!}$ for $0 \leq r \leq n$.

Explanation: The coefficients $C(n,0), C(n,1), \ldots, C(n,n)$ are called binomial coefficients. These are the same that were studied in Section 12.5.

Proof: The theorem will be proved by the principle of mathematical induction. Let $P(n)$ be the statement:

$$(a+b)^n = C(n,0)a^n + C(n,1)a^{n-1}b + \ldots + C(n,n-1)ab^{n-1} + C(n,n)b^n.$$

First, we verify the truth of $P(1)$

By taking $n = 1$, the statement $P(1)$ is

$$\begin{aligned}(a+b)^1 &= C(1,0)a^1 + C(1,1)b^1 \\ &= a + b.\end{aligned}$$

Thus $P(1)$ is true. Next suppose that $P(r)$ is true for some positive integer r. We shall prove that $P(r+1)$ is true. Now,

$= (a+b)\left[C(r,0)a^r + C(r,1)a^{r-1}b + \ldots + C(r,r-1)ab^{r-1} + C(r,r)b^r\right].$
(because $P(r)$ is assumed to be true).
$= C(r,0)a^{r+1} + C(r,1)a^r b + \ldots + C(r,r-1)a^2b^{r-1} + C(r,r)ab^r$
$+C(r,0)a^r b + C(r,1)a^{r-1}b^2 + \ldots + C(r,r-1)ab^r + C(r,r)b^{r+1}$
(by actual multiplication)
$= C(r,0)a^{r+1} + [C(r,1)+C(r,0)]\,a^r b + [C(r,2)+C(r,1)]\,a^{r-1}b^2 + \ldots +$
$[C(r,r-1)+C(r,r)]\,ab^r + C(r,r)b^{r+1}$
(by grouping the like terms)
$= C(r+1,0)a^{r+1} + C(r+1,1)a^r b + C(r+1,2)a^{r-1}b^2 + \ldots +$
$C(r+1,r)ab^r + C(r+1,r+1)b^{r+1}$

[by using (i) $C(r+1,0) = C(r,0)$, both being equal to 1;
(ii) $C(r,k) + C(r,k-1) = C(r+1,k)$
and (iii) $C(r+1,r+1) = C(r,r)$, both being equal to 1]

This proves that $P(r+1)$ is true if $P(r)$ is true. Therefore, by the principle of induction, $P(n)$ is true for all natural numbers n. This proves the binomial theorem.

An abbreviated form of the theorem

$$(a+b)^n = \sum_{r=0}^{n} C(n,r)a^{n-r}b^r.$$

Explanation: This sigma notation in the right side stands for

$$C(n,0)a^{n-0}b^0 + C(n,1)a^{n-1}b^1 + \ldots + C(n,n)a^{n-n}b^n$$

Observing that $b^0 = 1$ and $a^{n-n} = 1$, this is the same as the expansion given in the binomial theorem.

Explanation of binomial expansion: The expansion $C(n,0)a^n + C(n,1)a^{n-1}b + \ldots + C(n,n-1)ab^{n-1} + C(n,n)b^n$ obtained in the binomial theorem is called the binomial expansion of $(a+b)^n$. Though we have proved it, it still remains to explain why the combinatorial coefficients $C(n,r)$ appear in the expansion of $(a+b)^n$.
We know that $(a+b)^n = \underbrace{(a+b)(a+b)\ldots(a+b)}_{n\text{ times}}$
When we actually multiply these n factors, how do the terms look like? We see,

$$\begin{aligned}(a+b)(a+b) &= a.a + a.b + b.a + b.b\\ (a+b)(a+b)(a+b) &= (a.a + a.b + b.a + b.b)(a+b)\\ &= a.a.a + a.b.a + b.a.a + b.b.a\\ &\quad + a.a.b + a.b.b + b.a.b + b.b.b\end{aligned}$$

We observe that each term here corresponds to a selection of either a or b from each of the three factors of (a+b)(a+b)(a+b).

Choosing a from each one of them, we obtain the term $a.a.a$. Choosing a from the first, b from the second, and a from the third, we obtain the term $a.b.a$. Choosing b from the first, a from the second, and a from the third, we obtain the term $b.a.a$. and so on. There are $2^3 = 8$ such choices and that is why there are eight terms here (before regrouping). We can regroup and simplify the same as

$$a^3 + 3a^2b + 3ab^2 + b^3$$

This corresponds to the fact that there are 3 ways of choosing a from two of the three brackets and b from the other; there are 3 ways of choosing a from one of the three brackets and b from the remaining 2; etc.

More generally, in the expansion of $(a+b)^n$, if we want to choose b from r brackets and a from the remaining $(n-r)$ brackets, then there are $C(n,r)$ ways to do so. Therefore,

$$(a+b)^n = \sum_{r=0}^{n} C(n,r)a^{n-r}b^r$$

is explained. This can be considered as an alternative proof of the binomial theorem.

How to remember the binomial expansion: The following observations help us to remember the binomial expansion correctly:

1. The expansion of $(a+b)^n$ has $n+1$ terms. In other words, the number of terms is one more than the exponent.

2. In the successive terms of the expansion, the index of a goes on decreasing by unity, starting from n, then $n-1$, ... and ending with zero; on the contrary, the index of b goes on increasing by unity, starting from 0, then 1,... and ending with n.

3. In any term, the sum of indices of a and b is equal to n.

4. The binomial coefficients can be remembered by observing the following known as Pascal's Triangle.

Index of the binomial	*The binomial coefficients*
0	1
1	1 — 1
2	1 — 2 — 1
3	1 — 3 — 3 — 1
4	1 — 4 — 6 — 4 — 1
5	1 5 10 10 5 1

Here we note that each row is bounded by 1 on both sides. Any entry in a row is the sum of two entries in the preceding row, one on the immediate left and the other on the immediate right.

Some particular cases of binomial expansion

1. $(x-y)^n = C(n,0)x^n - C(n,1)x^{n-1}y + C(n,2)x^{n-2}y^2 - \ldots + (-1)^n C(n,n)y^n$.
   This is obtained by taking $a = x$ and $b = -y$ in the expansion of $(a+b)^n$.
   Note that the terms are alternately positive and negative.
2. $(1+x)^n = C(n,0) + C(n,1)x + C(n,2)x^2 + \ldots + C(n,n)x^n$
3. $(1-x)^n = C(n,0) - C(n,1)x + C(n,2)x^2 - \ldots + (-1)^n C(n,n)x^n$

Some special terms in the binomial expansion

1. In the expansion of $(a+b)^n$, the $(r+1)^{\text{th}}$ term is $C(n,r)a^{n-r}b^r$.
   This is called the *general term.*
2. In the expansion of $\left(x+\frac{1}{x}\right)^{2n}$, the $(n+1)^{\text{th}}$ term is the *middle term.* It is $C(2n,n)x^n.\frac{1}{x^n} = C(2n,n)$. This is called the *term independent of* x, or the *constant term.*
3. In the expansion of $(a+b)^n$, the middle term is the $\left(\frac{n}{2}+1\right)^{\text{th}}$ term if n is even. If n is odd, the two middle terms are $\left(\frac{n+1}{2}\right)^{\text{th}}$ and $\left(\frac{n+1}{2}+1\right)^{\text{th}}$

Example 13.1
Expand $(1-x+x^2)^4$

Solution
Let $y = -x + x^2$. Now,

$$\begin{aligned}
(1-x+x^2)^4 &= (1+y)^4 = 1 + C(4,1)y + C(4,2)y^2 + C(4,3)y^3 + y^4 \\
&= 1 + 4y + 6y^2 + 4y^3 + y^4 \\
&= 1 + 4(-x+x^2) + 6(-x+x^2)^2 + 4(-x+x^2)^3 + (-x+x^2)^4 \\
&= 1 + 4x(x-1) + 6x^2(x-1)^2 + 4x^3(x-1)^3 + x^4(x-1)^4 \\
&= 1 + 4x(x-1) + 6x^2(x^2-2x+1) + 4x^3(x^3-3x^2+3x-1) \\
&\quad + x^4(x^4-4x^3+6x^2-4x+1) \\
&= 1 + 4x^2 - 4x + 6x^4 - 12x^3 + 6x^2 + 4x^6 - 12x^5 + 12x^4 - 4x^3 + x^8 \\
&\quad - 4x^7 + 6x^6 - 4x^5 + x^4 \\
&= 1 - 4x + 10x^2 - 16x^3 + 19x^4 - 16x^5 + 10x^6 - 4x^7 + x^8
\end{aligned}$$

Example 13.2

Show that the middle term in the expansion of $(1+x)^{2n}$ is $\frac{1.3.5.\dots(2n-1)}{n!}2^n x^n$, where n is a positive integer.

Solution

There are $2n+1$ terms in the expansion of $(1+x)^{2n}$. The middle term is the $(n+1)^{\text{th}}$. It is $C(2n,n)1^{2n-n}x^n = C(2n,n)x^n$.

$$\begin{aligned}
\text{Now, } C(2n,n) &= \frac{(2n)!}{n!n!} = \frac{1.2.3\dots(2n-1).2n}{n!n!} \\
&= \frac{[1.3.5\dots(2n-1)]\,[2.4.6\dots(2n)]}{n!n!} \\
&= \frac{[1.3.5\dots(2n-1)]\,2^n.[1.2.3\dots n]}{n!n!} \\
&= \frac{[1.3.5\dots(2n-1)]\,2^n}{n!}
\end{aligned}$$

Therefore, the middle term is as stated in the problem.

Example 13.3

(i) Find the coefficient of x^{10} in the binomial expansion of $\left(2x^2 - \frac{3}{x}\right)^{11}$ where $x \neq 0$.

(ii) Prove also that there is no term involving x^6.

Solution

(i) The general term in this expansion is

$$C(11,r)(2x^2)^{11-r}\left(-\frac{3}{x}\right)^r$$

In this term, the exponent of x is $2(11-r)-r = 22-3r$. This will be equal to 10 exactly when $r=4$. The coefficient in that term is

$$C(11,4)2^{11-4}(-3)^4 = \frac{11\times10\times9\times8}{1\times2\times3\times4}\times2^7\times3^4 = 2^8.3^5.5.11$$

This is the required coefficient of x^{10}.

(ii) The general term is of the form $k.x^{22-3r}$. The index $22-3r$ *never takes the value* 6, for *any integer-value of* r. Therefore, there is no term involving r^6.

Example 13.4
If the coefficients of a^{r-1}, a^r, a^{r+1} in the binomial expansion of $(1+a)^n$ are in arithmetic progression, prove that $n^2 - n(4r+1) + 4r^2 - 2 = 0$.

Solution
The general term is $C(n,r)a^r$. The given assumption means that $C(n,r-1)$, $C(n,r)$ and $C(n,r+1)$, are in arithmetic progression. This means

$$C(n,r-1) + C(n,r+1) = 2C(n,r)$$
$$\therefore \frac{n!}{(r-1)!(n-r+1)!} + \frac{n!}{(r+1)!(n-r-1)!} = \frac{2 \times n!}{r!(n-r)!}$$
$$\frac{1}{(r-1)!(n-r-1)!}\left[\frac{1}{(n-r)(n-r+1)} + \frac{1}{r(r+1)}\right] = \frac{2}{(r-1)!(n-r-1)!r(n-r)}$$
$$\therefore \frac{1}{(n-r)(n-r+1)} + \frac{1}{r(r+1)} = \frac{2}{r(n-r)}$$

$$\begin{aligned}
\frac{r(r+1)+(n-r)(n-r+1)}{(n-r)(n-r+1)r(r+1)} &= \frac{2}{r(n-r)} \\
r(r+1)+(n-r)(n-r+1) &= 2(r+1)(n-r+1) \\
r^2+r+n^2-2nr+r^2+n-r &= 2\left[nr-r^2+n+1\right] \\
&= 2nr-2r^2+2n+2 \\
n^2-4nr-n+4r^2-2 &= 0 \\
n^2-n(4r+1)+4r^2-2 &= 0.
\end{aligned}$$

Example 13.5
Find the value of r if the coefficients of $(2r+4)^{th}$ and $(r-2)^{th}$ terms in the expansion of $(1+x)^{18}$ are equal.

Solution
We know that
$(1+x)^{18} = 1 + C(18,1)x + C(18,2)x^2 + \ldots + x^{18}$
Here, $(2r+4)^{th}$ term is $C(18,2r+3)x^{2r+3}$ and $(r-2)^{th}$ term is $C(18,r-3)x^{r-3}$
Therefore, either $2r+3 = r-3$ or $2r+3+r-3 = 18$
The former is not possible since r has to be positive. The latter gives $r = 6$.

EXERCISE 13.1

1. Expand $(1 + x +$

2. Find the term independent of x in the expansion of $\left(x - \frac{1}{x}\right)^{12}$
3. Find the value of $(\sqrt{2}+1)^5 + (\sqrt{2}-1)^5$.
4. Write the general term in the expansion of $(x^2 - y)^6$.
5. Expand $\left(x + \frac{1}{y}\right)^{11}$
6. What is the coefficient of x^5 in $(x+3)^6$?
7. In the binomial expansion of $(1+a)^{m+n}$, prove that the coefficients of a^m and a^n are equal.
8. Find the middle terms in the expansion of $\left(3x - \frac{x^3}{6}\right)^7$
9. Prove that the coefficient of x^n in $(1+x)^{2n}$ is twice the coefficient of x^n in $(1+x)^{2n-1}$.
10. In the binomial expansion of $(a+b)^n$, the coefficients of the fourth and thirteenth terms are equal to each other. Find n.
11. The coefficients of three consecutive terms in the expansion of $(1+a)^n$ are in the ratio 1:7:42. Find n.
12. The coefficients of $(r-1)^{\text{th}}$, r^{th} and $(r+1)^{\text{th}}$ terms in the expansion of $(x+1)^n$ are in the ratio 1:3:5. Find both n and r.
13. In the binomial expansion of $(1+x)^n$, the coefficients of the fifth, sixth and seventh terms are in arithmetic progression. Find all values of n for which this can happen.
14. Show that the coefficient of the middle term of $(1+x)^{2n}$ is equal to the sum of the coefficients of the two middle terms of $(1+x)^{2n-1}$

13.2 Some Applications of Binomial Theorem

In this section we study some applications of the theorem that we proved in section 13.1. The simplest of them is the straightforward application of the formula to compute some powers as in Example 13.6. A less trivial application is to prove some divisibility properties, as in Example 13.7. An important application is to prove some combinatorial identities, as shown in Examples 13.8, 13.9 and 13.11 below. In the next section, a more general form of binomial theorem will be applied to find the appropriate values of certain expressions.

Example 13.6

Using the binomial theorem, compute $(99)^5$.

Solution

In the binomial expansion of $(a+b)^n$, take $a = 100$, $b = -1$ and $n = 5$. We have:

$$(100-1)^5 = 100^5 - C(5,1)100^4.1 + C(5,2)100^3.1^2 - C(5,3)100^2.1^3 + C(5,4)100.1^4$$
$$- C(5,5)1^5$$

$$= 100^5 - 5 \times 100^4 + 10 \times 100^3 - 10 \times 100^2 + 5 \times 100 - 1$$
$$= 10010000500 - 500100001$$
$$= 9509900499$$

Thus $(99)^5 = 9509900499$

Example 13.7

If a and b are distinct integers, prove that $a^n - b^n$ is divisible by $a - b$, whenever n is a positive integer.

Solution

Because $a = a - b + b$, we have
$a^n = (a-b+b)^n = (a-b)^n + C(n,1)(a-b)^{n-1}b + \ldots + C(n,n)b^n$. The last term here is b^n. Bringing it to the left side,

$$a^n - b^n = (a-b)^n + C(n,1)(a-b)^{n-1}b + \ldots + C(n,n-1)(a-b)b^{n-1}.$$

In the right side, every term has $a - b$ as a factor. It follows that $a - b$ is a factor of $a^n - b^n$.

Example 13.8

Prove that the following hold for every natural number n:

(i) $C(n,0) + C(n,1) + \ldots + C(n,n) = 2^n$

(ii) $C(n,0) + 2C(n,1) + \ldots + 2^n C(n,n) = 3^n$

Solution

In the binomial expansion

$$(a+b)^n = C(n,0)a^n + C(n,1)a^{n-1}b + \ldots + C(n,n)b^n$$

First taking $a = 1 = b$, we obtain

$$(1+1)^n = C(n,0) + C(n,1) + \ldots + C(n,n).$$

This proves (i).

Next taking $a = 1$ and $b = 2$, we obtain

$$(1+2)^n = C(n,0) + C(n,1).2 + C(n,2)2^2 + \ldots + C(n,n)2^n.$$

This proves (ii).

Remark

The identity (i) has already been mentioned in §12.9. Presently, we have given an alternate proof. The identity (ii) has not been seen in the earlier chapters.

Example 13.9

Using the binomial theorem, prove the following identities:

(i) $C(n,0) + C(n,2) + C(n,4) + \ldots = 2^{n-1}$

(ii) $C(n,1) + C(n,3) + C(n,5) + \ldots = 2^{n-1}$

(iii) $C(n,0) + 3C(n,1) + 5C(n,2) + \ldots + (2n+1)C(n,n) = (n+1)2^n$

Explanation: (i) and (ii) means that the sum of the even binomial coefficients = the sum of the odd binomial coefficients = 2^{n-1} and hence it follows that the sum of all binomial coefficients in the expansion of $(a+b)^n$ is 2^n.

Solution

In the binomial expansion of $(a+b)^n$, taking $a=1$ and $b=-1$, we have
$(1-1)^n = C(n,0) - C(n,1) + C(n,2) - C(n,3) + \ldots$
Here the left side is zero. Therefore, we can rewrite this in the form
$C(n,0) + C(n,2) + C(n,4) + \ldots = C(n,1) + C(n,3) + C(n,5) + \ldots$
Let each of them be equal to x. Adding them

$C(n,0) + C(n,1) + C(n,2) + \ldots + C(n,n) = x + x = 2x$

We have already proved that the left side (the sum of all the binomial coefficients) is equal to 2^n (See Example 13.8). Thus $2^n = 2x$. Dividing both sides by 2, we get $x = 2^{n-1}$. This proves (i) and (ii) simultaneously.

Let $C(n,0) + 3C(n,1) + 5C(n,2) + \ldots + (2n+1)C(n,n) = x$ (13.1)

Using the formula, $C(n,r) = C(n,n-r)$, the same can be written as

$C(n,n) + 3C(n,n-1) + 5C(n,n-2) + \ldots + (2n+1)C(n,0) = x$

We write the terms in the reverse order, and obtain

$(2n+1)C(n,0) + (2n-1)C(n,1) + \ldots + 5C(n,n-2) + 3C(n,n-1) + C(n,n) = x$ (13.2)

We add (13.1) and (13.2) and obtain

$(2n+2)C(n,0) + (2n+2)C(n,1) + \ldots + (2n+2)C(n,n) = 2x.$

That is,

$(2n+2)\,[C(n,0) + C(n,1) + \ldots + C(n,n)] = 2x$

But we already know that the sum of the binomial coefficients is 2^n. Thus

$(2n+2)\cdot 2^n = 2x$

Dividing both sides by 2, $(n+1)2^n = x$
This proves, because of (13.1), that

$$C(n,0) + 3C(n,1) + 5C(n,2) + \ldots + (2n+1)C(n,n) = (n+1)2^n$$

Example 13.10
Find the coefficient of x^5 in the expansion of the product $(1+2x)^6(1-x)^7$.

Solution
We know

$$\begin{aligned}(1+2x)^6 &= 1 + C(6,1)(2x) + C(6,2)(2x)^2 + C(6,3)(2x)^3 + C(6,4)(2x)^4 \\ &\quad + C(6,5)(2x)^5 + C(6,6)(2x)^6 \\ &= 1 + 6(2x) + 15(4x^2) + 20(8x^3) + 15(16x^4) + 6(32x^5) + 64x^6 \\ &= 1 + 12x + 60x^2 + 160x^3 + 240x^4 + 192x^5 + 64x^6\end{aligned}$$

Also, $(1-x)^7$

$$\begin{aligned} &= 1 - C(7,1)x + C(7,2)x^2 - C(7,3)x^3 + C(7,4)x^4 - C(7,5)x^5 \\ &\quad + C(7,6)x^6 - C(7,7)x^7 \\ &= 1 - 7x + 21x^2 - 35x^3 + 35x^4 - 21x^5 + 7x^6 - x^7.\end{aligned}$$

We want to find the coefficient of x^5 in the product

$$(1+12x+60x^2+160x^3+240x^4+192x^5+64x^6)(1-7x+21x^2-35x^3+35x^4-21x^5+7x^6-x^7)$$

We need not carry out the full multiplication and write down the 56 terms. It is enough to observe which terms in the product involve x^5. They arise as

$$1.(-21x^5) + (12x)(35x^4) + (60x^2)(-35x^3) + (160x^3)(21x^2) + (240x^4)(-7x) + (192x^5).1$$

Explanation: When a term involving x^r is multiplied by a term involving x^{5-r}, we get a term involving x^5. Here r varies through 0,1,2,3,4,5.
$\therefore$ The coefficient of x^5 in the product is

$$(-21) + (12)(35) + (60)(-35) + (160)(21) + (240)(-7) + 192 = 171.$$

Example 13.11
If C_r denotes the binomial coefficient $C(n,r)$, prove that

$$C_0^2 + C_1^2 + \ldots + C_n^2 = \frac{(2n)!}{(n!)^2}$$

Solution

We know that

$$(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_nx^n$$

and

$$(x+1)^n = C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \ldots + C_n$$

Multiplying these,

$$\begin{aligned}(1+x)^n(x+1)^n &= (C_0 + C_1x + C_2x^2 + \ldots + C_nx^n)(C_0x^n + C_1x^{n-1} + C_2x^{n-2} \\ &\quad + \ldots + C_n)\end{aligned}$$

If in the right side, the multiplication is actually carried out, the coefficient of x^n is

$$C_0^2 + C_1^2 + \ldots + C_n^2$$

This should be equal to the coefficient of x^n in the left side. But the left side can be rewritten as $(1+x)^{2n}$. Here the coefficient of x^n is $C(2n, n) = \dfrac{(2n)!}{(n!)^2}$. This proves the required identity.

Example 13.12

Using Binomial Theorem, expand $(a+b)^6 - (a-b)^6$. Hence find the value of $(\sqrt{2}+1)^6 - (\sqrt{2}-1)^6$

Solution

$$\begin{aligned}(a+b)^6 - (a-b)^6 &= C(6,0)a^6 + C(6,1)a^5b + C(6,2)a^4b^2 \\ &\quad + C(6,3)a^3b^3 + C(6,4)a^2b^4 + C(6,5)ab^5 + C(6,6)b^6 \\ &\quad - C(6,0)a^6 + C(6,1)a^5b - C(6,2)a^4b^2 \\ &\quad + C(6,3)a^3b^3 - C(6,4)a^2b^4 + C(6,5)ab^5 - C(6,6)b^6 \\ &= 2\left[C(6,1)a^5b + C(6,3)a^3b^3 + C(6,5)ab^5\right] \text{ (Since other terms cancel)} \\ &= 2\left[6a^5b + 20a^3b^3 + 6ab^5\right] \\ &= 4ab\left[3a^4 + 10a^2b^2 + 3b^4\right]\end{aligned}$$

Taking $a = \sqrt{2}$ and $b = 1$, we have

$$\begin{aligned}(\sqrt{2}+1)^6 - (\sqrt{2}-1)^6 &= 4\sqrt{2}\left[3(\sqrt{2})^4 + 10(\sqrt{2})^2 + 3\right] \\ &= 4\sqrt{2}[12 + 20 + 3] = 140\sqrt{2}.\end{aligned}$$

EXERCISE 13.2

1. Evaluate $(999)^5$ using the binomial theorem.
2. Find the value of $(102)^6$ by means of the binomial theorem.
3. Evaluate $(\sqrt{3}+1)^5 - (\sqrt{3}-1)^5$.
4. Find $(99)^4$, using the binomial theorem.
5. Write down the binomial expansion of $(1+x)^{n+1}$, when $x = 8$. Deduce that $9^{n+1} - 8n - 9$ is divisible by 64, whenever n is a positive integer.
6. Using binomial theorem, prove that $6^n - 5n$ always leaves the remainder 1 when divided by 25.
7. Prove that $\sum_{r=0}^{n} 3^r C(n,r) = 4^n$.

   Prove the following identities 8 to 13 using binomial theorem or otherwise. Here C_r denotes $C(n,r)$.

8. $C_0 + C_2 + C_4 + \ldots = 2^{n-1}$
9. $\frac{C_1}{C_0} + 2\frac{C_2}{C_1} + \ldots + n\frac{C_n}{C_{n-1}} = \frac{n(n+1)}{2}$
10. $(C_0 + C_1)(C_1 + C_2)\ldots(C_{n-1} + C_n) = \frac{C_0C_1\ldots C_{n-1}(n+1)^n}{n!}$
11. $C_0 + 2C_1 + 3C_2 + \ldots + (n+1)C_n = 2^n + n.2^{n-1}$
12. $C_0C_1 + C_1C_2 + \ldots + C_{n-1}C_n = \frac{2^n.n.1.3.5.\ldots.(2n-1)}{(n+1)!}$
13. $C_1 + 2C_2 + 3C_3 + \ldots + nC_n = n.2^{n-1}$
14. Find the coefficient of x^{n-r} in the expansion of $(x+1)^n(1+x)^n$. Deduce that
$$C_0C_r + C_1C_{r+1} + \ldots + C_{n-r}C_n = \frac{(2n)!}{(n-r)!(n+r)!}.$$
15. Find the coefficient of x^4 in the expansion of $(1+x)^n(1-x)^n$. Deduce that
$$C_2 = C_0C_4 - C_1C_3 + C_2C_2 - C_3C_1 + C_4C_0.$$

13.3 Binomial Theorem for any Index

In this section we state a more general binomial theorem, in which the index is not necessarily a whole number. We also give an application of this theorem to find approximate values of certain quantities.

We know the formula

$$(1+x)^n = C(n,0) + C(n,1)x + \ldots + C(n,n)x^n$$

Here n may be any non-negative integer. We ask: Does the same formula hold, when n is a fraction or a negative number? Immediately, we note that the coefficients $C(n,r)$ do not make sense, when n is not a whole number. This difficulty may be overcome, by writing,

$$\frac{n(n-1)\dots(n-r+1)}{1.2.\dots r}$$

in place of $C(n,r)$. This of course makes sense even when n is not a whole number.

We now state, without proof, the more general theorem, in which the index is not a whole number.

Theorem 13.1

The formula

$$(1+x)^m = 1 + m.x + \frac{m(m-1)}{1.2}x^2 + \frac{m(m-1)(m-2)}{1.2.3}.x^3 + \dots$$

holds whenever $|x| < 1$.

Remark

1. Note carefully the condition that $|x|$ should be less than 1. This extra condition is unnecessary, if n is a whole number. Let us see how this formula fails to hold when $x = 1$ and $m = -1$ (because $|x|$ is not less than 1). Left side is $(1+1)^{-1} = \frac{1}{2}$.

   Right side is $1 + (-1).1 + \frac{(-1)(-2)}{1.2}1^2 + \dots = 1 - 1 + 1 - 1 + \dots$. In the series $1 - 1 + 1 - 1 + \dots$ by taking first one, two, three, ...terms and adding we get alternately $1, 0, 1, 0, \dots$, so that these sums can never be near $\frac{1}{2}$.

2. Note that there are infinite number of terms in the expansion of $(1+x)^m$, when m is a negative integer or a fraction.

Consider

$$\begin{aligned}(a+b)^m &= \left[a(1+\frac{b}{a})\right]^m = a^m\left(1+\frac{b}{a}\right)^m \\ &= a^m\left[1 + m.\frac{b}{a} + \frac{m(m-1)}{1.2}\left(\frac{b}{a}\right)^2 + \dots\right] \\ &= a^m + m.a^{m-1}b + \frac{m(m-1)}{1.2}a^{m-2}b^2 + \dots\end{aligned}$$

This expansion is valid when $\left|\frac{b}{a}\right| < 1$ or equivalently when $|b| < |a|$. Thus we have

Theorem 13.2
The formula

$$(a+b)^m = a^m + m.a^{m-1}b + \frac{m(m-1)}{1.2}.a^{m-2}b^2 + \dots$$

holds whenever $|b| < |a|$

Remark

These two theorems are known by the name 'binomial theorem for general index'. This expansion is also known as binomial series.
The term $\frac{m(m-1)(m-2)\dots(m-r+1)}{1.2\dots r}x^r$ is called the general term in the binomial expansion of $(1+x)^m$.
Similarly the general term in the expansion of $(a+b)^m$ is

$$\frac{m(m-1)\dots(m-r+1)}{1.2\dots r}.a^{m-r}b^r$$

Some inportant particular cases of the binomial series

1. Taking $m = -1$ in the expansion of $(1+x)^m$, we have:

$$\frac{1}{1+x} = 1 + (-1)x + \frac{(-1)(-2)}{1.2}.x^2 + \frac{(-1)(-2)(-3)}{1.2.3}.x^3 + \dots = 1 - x + x^2 - x^3 + \dots$$

2. Taking $x = -a$ in the above formula, we have

$$\frac{1}{1-a} = 1 + a + a^2 + a^3 + \dots$$

3. Taking $m = -2$ in the expansion of $(1+x)^m$, we have

$$\frac{1}{(1+x)^2} = 1+(-2)x+\frac{(-2)(-3)}{1.2}x^2+\frac{(-2)(-3)(-4)}{1.2.3}.x^3+\dots = 1-2x+3x^2-4x^3+\dots$$

4. Taking $x = -a$ in the formula 3, we have

$$\frac{1}{(1-a)^2} = 1 + 2a + 3a^2 + 4a^3 + \dots$$

In all these formulae, we assume $|x| < 1$. It is good to remember these formulae.

Let $|x| < 1$. Then

$$\begin{aligned}
\frac{1}{1+x} &= 1 - x + x^2 - x^3 + \dots \\
\frac{1}{1-x} &= 1 + x + x^2 + x^3 + \dots \\
\frac{1}{(1+x)^2} &= 1 - 2x + 3x^2 - 4x^3 + \dots \\
\frac{1}{(1-x)^2} &= 1 + 2x + 3x^2 + 4x^3 + \dots
\end{aligned}$$

Differences between the binomial theorems for positive integer exponent and for general exponent

1. *Finite versus infinite*

   In the expansion of $(1+x)^m$ there are infinitely many terms, when m is a general exponent. When m is a positive integer, there are only $m+1$ terms.

2. *All x versus some x*

   When m is a general exponent, the expansion of $(1+x)^m$ is valid under the extra assumption $|x| < 1$. When m is a positive integer, the expansion is valid for all values of x.

3. *Exact versus approximate*

   The actual calculation of $(1+x)^m$ through the binomial expansion can be done only up to a finite number of terms. The value calculated upto these finite number of terms will only be an approximate value of $(1+x)^m$, and not its exact value, when m is general. But when m is a positive integer, the exact value of $(1+x)^m$ is obtained by the $(m+1)$ terms of the binomial expansion.

4. *The notation $C(n,r)$*
In the expansion of $(1+x)^n$, the coefficients should not involve the notations $C(n,0), C(n,1)$ etc. unless n is a whole number. They have to be explicitly written in the form $\frac{n(n-1)\ldots}{1.2.\ldots}$

Explanation: If the binomial expansion for $(1+x)^m$ is to be valid, there has to be a restriction either on x or on m. When we restrict m to be a whole number, it is true for all values of x. When $-1 < x < 1$, it is true for all values of m.

Note

A word of caution is necessary when we use some values of m such as $m = \frac{1}{2}$. According to the theorem

$$(1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}x^2 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}x^3 + \ldots$$

i.e.

$$(1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x - \frac{x^2}{8} + \frac{x^3}{16} \ldots$$

Now the L.H.S. being a square root of $(1+x)$ has apparently two values (a positive value and a negative value). Which of these is represented by the R.H.S.? It can be shown that the R.H.S. represents only one value viz. the "positive" square root of $(1+x)$, which is positive for $|x| < 1$.

Example 13.13

Prove that the coefficient of y^n in the expansion of $\frac{(1+y)^2}{(1-y)^2}$ is $4n$, for each $n = 1, 2, 3, \ldots$

Solution

$$\begin{aligned}(1+y)^2 &= 1 + 2y + y^2\\ \frac{1}{(1-y)^2} &= (1-y)^{-2} = 1 + 2y + 3y^2 + \ldots\end{aligned}$$

In the product $(1+2y+y^2)(1+2y+3y^2+4y^3+\ldots)$, the terms involving y^n are $1.(n+1)y^n + (2y)(ny^{n-1}) + (y^2)\left\{(n-1)y^{n-2}\right\}$, if $n \geq 2$. The coefficient of y^n is

$$1.(n+1) + 2.n + 1.(n-1) = 4n$$

We directly verify that the coefficient of y is 4. Therefore, the result is true for $n = 1$ also.

Example 13.14
Expand $\frac{1}{\sqrt[3]{6-3x}}$

Solution

$$\frac{1}{\sqrt[3]{6-3x}} = (6-3x)^{\frac{-1}{3}}$$

Its binomial expansion is valid when $|3x| < 6$, that is, when $|x| < 2$. The series is

$$6^{\frac{-1}{3}} + \left(-\frac{1}{3}\right) 6^{(\frac{-1}{3}-1)}(-3x) + \frac{\left(\frac{-1}{3}\right)\left(\frac{-1}{3}-1\right)}{1\cdot 2}.6^{(\frac{-1}{3}-2)}.(-3x)^2 + \ldots$$

$$= 6^{\frac{-1}{3}} + 6^{\frac{-4}{3}}x + 2\cdot 6^{\frac{-7}{3}}x^2 + \ldots$$
$$= 6^{\frac{-1}{3}}\left[1 + \frac{x}{6} + \frac{2x^2}{6^2} + \ldots\right]$$

Alternative Solution

$$\begin{aligned}(6-3x)^{\frac{-1}{3}} &= \left[6\left(1-\frac{x}{2}\right)\right]^{\frac{-1}{3}} \\ &= 6^{\frac{-1}{3}}\left(1-\frac{x}{2}\right)^{\frac{-1}{3}} \\ &= 6^{\frac{-1}{3}}\left[1 + \left(\frac{1}{3}\right)\left(\frac{x}{2}\right) + \frac{\left(\frac{-1}{3}\right)\left(\frac{-1}{3}-1\right)}{1\cdot 2}\left(\frac{-x}{2}\right)^2 + \ldots\right] \\ &= 6^{\frac{-1}{3}}\left[1 + \frac{x}{6} + \frac{x^2}{18} + \ldots\right]\end{aligned}$$

Example 13.15
Comment on the binomial series for $(1+x)^m$ when $x = -2$ and $m = -1$.

Solution
When $x = -2$, and $m = -1$, $(1+x)^m = (1-2)^{-1} = \frac{1}{-1} = -1$
The series would be

$$1 + (-1)(-2) + \frac{(-1)(-2)}{1\cdot 2}(-2)^2 + \frac{(-1)(-2)(-3)}{1\cdot 2\cdot 3}(-2)^3 + \ldots$$
$$= 1 + 2 + 2^2 + 2^3 + \ldots$$

It is clearly seen that this series does not have the required sum -1.

Thus the binomial series expansion is found to be invalid in this example. This does not contradict the binomial theorem, because the condition $|x| < 1$ fails here.

Example 13.16

(a) Write the first four terms in the expansion of $\dfrac{1}{(4-5x^2)^{\frac{1}{2}}}$

(b) For what values of x is this expansion valid?

(c) What is the general term?

Solution

(a)
$$\frac{1}{(4-5x^2)^{\frac{1}{2}}} = (4-5x^2)^{\frac{-1}{2}}$$
$$= \left[4\left(1-\frac{5x^2}{4}\right)\right]^{\frac{-1}{2}} = 4^{\frac{-1}{2}}\left(1-\frac{5x^2}{4}\right)^{\frac{-1}{2}}$$
$$= \frac{1}{2}\left[1+\left(-\frac{1}{2}\right)\left(\frac{-5x^2}{4}\right)+\right.$$
$$\left.\frac{\left(\frac{-1}{2}\right)\left(\frac{-1}{2}-1\right)}{1.2}\left(\frac{-5x^2}{4}\right)^2+\frac{\left(\frac{-1}{2}\right)\left(\frac{-1}{2}-1\right)\left(\frac{-1}{2}-2\right)}{1.2.3}\left(\frac{-5x^2}{4}\right)^3+\ldots\right]$$
$$= \frac{1}{2}\left[1+\frac{5x^2}{8}+\frac{3}{8}.\frac{25}{16}.x^4+\frac{5}{16}.\frac{125}{64}x^6+\ldots\right]$$

The first four terms are

$$\frac{1}{2}, \frac{5}{16}x^2, \frac{75}{256}x^4, \frac{625}{2048}x^6$$

(b) The expansion is valid when $|5x^2| < 4$. This happens when x lies between $\frac{-2}{\sqrt{5}}$ and $\frac{2}{\sqrt{5}}$.

(c) The general term is

$$\frac{1}{2}.\frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\ldots\left(-\frac{1}{2}-r+1\right)}{1.2.3\ldots r}\left(\frac{-5x^2}{4}\right)^r$$

Example 13.17
Find the cube root of 127 upto four places of decimals.

Solution

$$
\begin{aligned}
(127)^{\frac{1}{3}} &= (125+2)^{\frac{1}{3}} = \left[125\left(1+\frac{2}{125}\right)\right]^{\frac{1}{3}} \\
&= 125^{\frac{1}{3}}\left(1+\frac{2}{125}\right)^{\frac{1}{3}} \\
&= 5\left[1+\left(\frac{1}{3}\right)\left(\frac{2}{125}\right)+\frac{\left(\frac{1}{3}\right)\left(\frac{1}{3}-1\right)}{1.2}\left(\frac{2}{125}\right)^2+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)}{1.2.3}\left(\frac{2}{125}\right)^3+\right. \\
&= 5\left[1+\frac{2}{375}+\left(-\frac{1}{9}\right)\left(\frac{2}{125}\right)^2+\frac{5}{81}\left(\frac{2}{125}\right)^3+\ldots\right] \\
&= 5\left[1+\frac{16}{3000}-\frac{4}{9}.\frac{64}{10^6}+\frac{80}{81}.\frac{256}{10^9}+\ldots\right] \\
&= 5\left[1+0.0053-0.0000284+0.000000253-\ldots\right] \\
&= 5\left[1.005300253-0.0000284\right] \\
&= 5\times 1.005271853 \\
&= 5.0264 \text{ upto four places of decimals.}
\end{aligned}
$$

This expansion is valid because $\left|\frac{2}{125}\right| < 1$.

We have omitted the other terms because higher powers of $\frac{2}{125}$ do not contribute to the first four decimal places.

Example 13.18
If x is very small in magnitude when compared with a, show that

$$(\frac{a}{a+x})^{\frac{1}{2}}+(\frac{a}{a-x})^{\frac{1}{2}}=2+\frac{3x^2}{4a^2} \text{ nearly.}$$

Solution

$$
\begin{aligned}
\left(\frac{a}{a+x}\right)^{\frac{1}{2}}+\left(\frac{a}{a-x}\right)^{\frac{1}{2}} &= \left(\frac{1}{1+\frac{x}{a}}\right)^{\frac{1}{2}}+\left(\frac{1}{1-\frac{x}{a}}\right)^{\frac{1}{2}} \\
&= \left(1+\frac{x}{a}\right)^{\frac{-1}{2}}+\left(1-\frac{x}{a}\right)^{\frac{-1}{2}}
\end{aligned}
$$

Our assumption implies that $|x| < |a|$ and therefore $\left|\frac{x}{a}\right| < 1$. Hence, the binomial series expansion is valid. Also, the terms involving $(\frac{x}{a})^3, (\frac{x}{a})^4, \dots$ can be ignored since they will be negligibly small.
Therefore,
$(1+\frac{x}{a})^{\frac{-1}{2}} + (1-\frac{x}{a})^{\frac{-1}{2}}$ is approximately equal to

$$1+\left(-\frac{1}{2}\right)\left(\frac{x}{a}\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{1.2}\left(\frac{x}{a}\right)^2+1+\left(-\frac{1}{2}\right)\left(-\frac{x}{a}\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{1.2}\left(\frac{-x}{a}\right)^2$$

$$= \left[1-\frac{x}{2a}+\frac{3x^2}{8a^2}\right]+\left[1+\frac{x}{2a}+\frac{3x^2}{8a^2}\right]$$
$$= 2+\frac{3x^2}{4a^2}$$

EXERCISE 13.3

1. Write down the binomial series for $\frac{1}{\sqrt{5+4x}}$ where $x < \frac{5}{4}$.
2. Expand $\frac{1}{(4-3x^2)^{\frac{1}{3}}}$ to four terms. For what values of x is the expansion valid?
3. Find the coefficient of x^6 in the expansion of $(1-2x)^{\frac{-5}{2}}$.
4. Prove that the coefficient of x^r in the expansion of $(1-4x)^{\frac{-1}{2}}$ is $\frac{(2r)!}{(r!)^2}$.
5. Assuming x to be so small that x^2 and higher powers of x can be neglected, show that $\frac{\left[1+\frac{3}{4}x\right]^{-4}[16-3x]^{\frac{1}{2}}}{(8+x)^{\frac{2}{3}}}$ is approximately equal to $1-\frac{305}{96}x$.
6. Prove that $\sqrt[3]{126} = 5.01330$ to five decimal places.
7. If $\frac{(1-3x)^{\frac{1}{2}}+(1-x)^{\frac{5}{3}}}{\sqrt{4-x}}$ is approximately equal to $a+bx$ for all small values of x, find a and b.
8. If p is nearly equal to q and $n > 1$, show that $\frac{(n+1)p+(n-1)q}{(n-1)p+(n+1)q} = (\frac{p}{q})^{\frac{1}{n}}$. Hence, find the approximate value of $(\frac{99}{101})^{\frac{1}{6}}$.
9. Find $(0.98)^{-3}$ upto two decimal places.

10. If all the coefficients of $(a+bx)^{-2}$ are positive, prove that a and b are of the opposite sign.
11. If the binomial expansion of $(a+bx)^{-2}$ is $\frac{1}{4} - 3x + \ldots$, find the values of a and b.
12. Find the two values of the rational exponent m such that in the binomial expansion of $(1-x)^m$, the coefficient of x^2 is 3.

MISCELLANEOUS EXERCISE ON CHAPTER 13

1. Find the coefficient of x^n in the product expansion of $(x+1)^n(1+x)^{-2}$. Use this to prove

$$C_o - 2C_1 + 3C_2 + \ldots + (-1)^n(n+1)C_n = 0$$

where C_r means $C(n,r)$.

2. If three consecutive coefficients in the expansion of $(1+x)^n$ are in the ratio 6:33:110, determine n.
3. If $x = 0.001$, prove that

$$\frac{(1-2x)^{\frac{2}{3}}(4+5x)^{\frac{3}{2}}}{\sqrt{1-x}} = 8.01 \text{ upto two decimal places.}$$

4. In the binomial expansion of $(1+x)^m$, the third term is $\frac{-1}{8}x^2$. Find the rational exponent m and hence write the fourth term.
5. If x is numerically so small that x^2 and higher powers of x may be neglected, then prove that

$$\frac{(1-2x)^{\frac{2}{3}}(4+5x)^{\frac{3}{2}}}{\sqrt{1-x}} = 8 + \frac{25x}{3}$$

6. Write first three terms in the expansion of $\frac{2+x}{(3-2x)^2}$.
7. Find the coefficient of x^4 in the expansion of $\left[\frac{1-x}{1+x}\right]^2$
8. Show that when x is numerically small, $\sqrt{x^2+4} - \sqrt{x^2+1} = 1 - \frac{x^2}{4} + \frac{7x^4}{64}$.

CHAPTER 14

Exponential and Logarithmic Series

In this chapter, we are going to introduce two important series called exponential series and logarithmic series. You may recall that you have already studied in earlier chapters, two other important kinds of series, namely geometric series and binomial series. The following from earlier chapters will be made use of:

Geometric series

Formula for the sum of infinite geometric series

Factorial of a non-negative integer

Binomial series expansion for $(1+x)^y$

The inequality $2^{n-1} \leq n!$ for all positive integers

The complex numbers, their real and imaginary parts

The formula $e^{i\theta} = \cos\theta + i\sin\theta$

The combinatorial coefficients $C(n, r)$, etc.

The results of this chapter will be useful for summation of some infinite series, and for the introduction and study of some important functions in higher classes.

14.1 Exponential Series

Consider the series of numbers

$$1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\cdots \qquad (i)$$

The sum of the series given in (i) is denoted by the number e.

Let us estimate the value of number e.

Since every term of the series (e) is positive, it is clear that its sum is also positive.

Consider the two sums

$$1+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\cdots+\frac{1}{n!}+\cdots \qquad (ii)$$

$$\text{and } \frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\cdots+\frac{1}{2^{n-1}}+\cdots \qquad (iii)$$

Oberseve that,

$$\frac{1}{3!}=\frac{1}{6} \text{ and } \frac{1}{2^2}=\frac{1}{4}$$

$$\therefore \frac{1}{3!}<\frac{1}{2^2}$$

$$\frac{1}{4!}=\frac{1}{24} \text{ and } \frac{1}{2^3}=\frac{1}{8}$$

$$\therefore \frac{1}{4!}<\frac{1}{2^3}$$

$$\frac{1}{5!}=\frac{1}{120} \text{ and } \frac{1}{2^4}=\frac{1}{16}$$

$$\therefore \frac{1}{5!}<\frac{1}{2^4}$$

Therefore, by analogy, we can say that

$$\frac{1}{n!}<\frac{1}{2^{n-1}}$$

We can see that each term in (*ii*) excluding the first term is less than its corresponding term in (*iii*)

$$\therefore \left(\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\cdots+\frac{1}{n!}+\cdots\right)<\left(\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\cdots+\frac{1}{2^{n-1}}+\cdots\right) \qquad (iv)$$

Adding $\left(1+\frac{1}{1!}+\frac{1}{2!}\right)$ on both sides of (*iv*) we get,

$$\left(1+\frac{1}{1!}+\frac{1}{2!}\right)+\left(\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\cdots+\frac{1}{n!}+\cdots\right)$$

$$<\left\{\left(1+\frac{1}{1!}+\frac{1}{2!}\right)+\left(\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\cdots+\frac{1}{2^{n-1}}+\cdots\right)\right\} \qquad (v)$$

$$=\left\{1+\left(1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\cdots+\frac{1}{2^{n-1}}+\cdots\right)\right\}$$

$$=1+\frac{1}{1-\frac{1}{2}}=1+2$$

$$=3$$

Left hand side of (*v*) represents the series (*i*)

$$\therefore e<3$$

Next, we consider the more general series $1+\frac{x}{1!}+\frac{x^2}{2!}+\ldots+\frac{x^n}{n!}+\ldots$ This is called the exponential series. When we put $x=1$ here, we obtain the series for e.

Notation: We denote the sum of this series by e^x. We shall prove the following results in the higher classes.

1. If x is a rational number, then e^x denotes both the sum of series $1+\frac{x}{1!}+\frac{x^2}{2!}+\ldots$ and the number obtained by raising the number e to the power x. But there is no confusion, because both these are the same. Even for irrational x, the same remark holds, but you have not studied raising a number to an irrational power.

2. Even for complex number x, one can prove that the series $1+\frac{x}{1!}+\frac{x^2}{2!}+\ldots$ has a sum and we donote it by e^x.

3. $e^{x+y}=e^x.e^y$ holds for all x, y. [This law of indices, you already know when x and y are rational.]

4. When θ is a real number, $i\theta$ is a complex number.

   Then $e^{i\theta}=\cos\theta+i\sin\theta$ holds. Thus $\cos\theta$ is the real part of $e^{i\theta}$. If we collect those terms in the series for $e^{i\theta}$ not involving i, we obtain

   $\cos\theta=1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}-\ldots$

   Similarly, looking at the imaginary part,

   $$\sin\theta=\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}-\ldots$$

   You may write down the series for $e^{i\theta}$, simplify the terms using $i^2=-1$, and obtain the above series of $\cos\theta$ and $\sin\theta$.

5. The number e is an irrational number.

We do not prove the above results here. But sometimes we make use of some of them in the problems below.

Some Particular Cases

The exponential series is

$$e^x=1+\frac{x}{1!}+\frac{x^2}{2!}+\ldots$$

1. When $x=0$, this becomes

   $$e^0=1+0+0+\ldots=1$$

   This is in conformity with the known fact that any non-zero number, when raised to the power zero, is 1.

2. When $x=1$, this becomes

   $$e^1=1+\frac{1}{1!}+\frac{1}{2!}+\ldots$$

   Since e^1 is same as e, this coincides with our meaning of the number e.

3. When $x = 2$, this becomes

$$e^2 = 1 + \frac{2}{1!} + \frac{2^2}{2!} + \dots$$

This means that the sum of the series on the right side is the same as the square of the number e, as already remarked.

4. When $x = -1$, this becomes

$$e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots$$

The sum of this series is the reciprocal of e.

5. For any number x,

$$e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots$$

This is obtained by substituting $-x$ for x in the exponential series. Note that the terms have alternately positive and negative signs in this series.

6. Adding the two series term by term, we get,

since, $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \dots$ and $e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$, therefore

$$\begin{aligned} e^x + e^{-x} &= (1+1) + \left(\frac{x}{1!} - \frac{x}{1!}\right) + \left(\frac{x^2}{2!} + \frac{x^2}{2!}\right) + \left(\frac{x^3}{3!} - \frac{x^3}{3!}\right) + \left(\frac{x^4}{4!} + \frac{x^4}{4!}\right) \dots \\ &= 2 + 0 + 2.\frac{x^2}{2!} + 0 + 2.\frac{x^4}{4!} \dots \\ &= 2\left[1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots\right] \end{aligned}$$

7. Similarly, one obtains a series expansion for $e^x - e^{-x}$

Example 14.1

(i) Define the number e

(ii) Prove that its value lies between 2 and 3

(iii) Show that $2.7 < e < 3$

Solution

(i) and (ii) have been done in the text above.

(iii) The first five terms in the series for e add up to $1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}=1+1+\frac{1}{2}+\frac{1}{6}+\frac{1}{24}>2.7$.
Since $e<3$ is already known, we have $2.7<e<3$.

Example 14.2

Prove that the coefficient of x^{10} in the series of e^{2x} is $\frac{4}{14175}$

Solution

We have the exponential series

$$e^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+\dots$$

Substituting $2x$ wherever x occurs,

$$e^{2x} = 1+\frac{2x}{1!}+\frac{(2x)^2}{2!}+\frac{(2x)^3}{3!}+\dots$$

Here, the coefficient of x^{10} is to be taken from the term $\frac{(2x)^{10}}{10!}$. The required coefficient is

$$\begin{aligned}\frac{2^{10}}{10!} &= \frac{2\times2\times2\times2\times2\times2\times2\times2\times2\times2}{1\times2\times3\times4\times5\times6\times7\times8\times9\times10}\\ &= \frac{2\times2\times2\times2}{3\times5\times6\times7\times9\times10}\\ &= \frac{2\times2}{3\times5\times3\times7\times9\times5}=\frac{4}{14175}\end{aligned}$$

Example 14.3

Sum the series $\sum_{n=2}^{\infty} C(n,2)\frac{3^{n-2}}{n!}$

Solution

Note that the summation is from $n=2$ onwards. Here, n does not take the value 1 because $C(1,2)$ is not defined by us. Here, the general term is

$$\begin{aligned}\frac{C(n,2)3^{n-2}}{n!} &= \frac{n(n-1)}{1.2}\cdot\frac{3^{n-2}}{n!}\\ &= \frac{3^{n-2}}{2.(n-2)!} \qquad \text{[Since the factors } n \text{ and } n-1 \text{ cancel out]} \qquad (14.1)\end{aligned}$$

Therefore the series is $\frac{1}{2}\left[1+\frac{3}{1!}+\frac{3^2}{2!}+\dots\right]$

Its sum is $\frac{1}{2}e^3$.

Example 14.4

Find the coefficient of x^2 in the expansion of e^{2x+3} as a series in powers of x.

Solution

$$e^{2x+3} = 1 + \frac{2x+3}{1!} + \frac{(2x+3)^2}{2!} + \ldots,$$

This is a series in powers of $(2x+3)$, and not of x. Here, the general term is $\frac{(2x+3)^n}{n!}$. This can be expanded by the Binomial Theorem as

$$\frac{1}{n!}\left[3^n + C(n,1)3^{n-1}(2x) + C(n,2)3^{n-2}(2x)^2 + \ldots + (2x)^n\right]$$

Here, the coefficient of x^2 is $\frac{C(n,2)3^{n-2}2^2}{n!}$. Therefore, the coefficient of x^2 in the whole series is

$$\sum_{n=2}^{\infty} \frac{C(n,2)3^{n-2}2^2}{n!}.$$

Proceeding as in Example 14.3, we find that this sum is $2e^3$. Thus $2e^3$ is the coefficient of x^2 in the expansion of e^{2x+3}.

Second Method: We may use the already stated result,

$$e^x.e^y = e^{x+y}$$

Now, $e^{2x+3} = e^{2x}.e^3 = e^3\left[1 + \frac{2x}{1!} + \frac{(2x)^2}{2!} + \ldots\right]$

Here, the coefficient of x^2 is $\frac{e^3.2^2}{2!} = 2e^3$

Example 14.5

Sum the series $1 + \frac{2^3}{2!} + \frac{3^3}{3!} + \frac{4^3}{4!} + \ldots$

[Note that this is different from the exponential series $1 + \frac{3}{1!} + \frac{3^2}{2!} + \frac{3^3}{3!} + \frac{3^4}{4!} + \ldots$]

Here the general term is $\frac{n^3}{n!}$.

Cancelling the common factor n both in the numerator and the denominator, this can be simplified as $\frac{n^2}{(n-1)!}$. Now the numerator n^2 can be written as

$$n^2 = (n-1)(n-2) + 3(n-1) + 1$$

Explanation: The form $A_0 + A_1(n-1) + A_2(n-1)(n-2) + \dots$ is helpful, as seen below. Here $A_0 = 1, A_1 = 3, A_2 = 1, A_3 = 0$, etc. Thus the general term is

$$\begin{aligned}\frac{n^2}{(n-1)!} &= \frac{(n-1)(n-2)}{(n-1)!} + \frac{3(n-1)}{(n-1)!} + \frac{1}{(n-1)!} \\ &= \frac{1}{(n-3)!} + \frac{3}{(n-2)!} + \frac{1}{(n-1)!}\end{aligned}$$

[A warning here: This makes sense only when n is at least 3. Otherwise, we come across the factorials of negative numbers.]
Therefore, we write the given series as

$$1 + \frac{2^3}{2!} + \left(\frac{1}{0!} + \frac{3}{1!} + \frac{1}{2!}\right) + \left(\frac{1}{1!} + \frac{3}{2!} + \frac{1}{3!}\right) + \dots$$

Regrouping the terms, we write this as

$$\left(1 + \frac{2^3}{2!}\right) + \left(\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \dots\right) + \left(\frac{3}{1!} + \frac{3}{2!} + \frac{3}{3!} + \dots\right) + \left(\frac{1}{2!} + \frac{1}{3!} + \dots\right)$$

The three series in the last three brackets can be summed up easily using the definition of e. The first of them is exactly e. The second is

$$3\left(\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots\right)$$

which is

$$3\left[-1 + \left(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots\right)\right]$$

and its sum is $3(-1+e)$.
The last is a

$$\left[\left(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots\right) - \left(1 + \frac{1}{1!}\right)\right]$$

and its sum is $e - (1+1) = e - 2$.
Thus the required answer is

$$\begin{aligned}&1 + \frac{2^3}{2!} + e + 3(e-1) + e - 2 \\ &= 1 + 4 + e + 3e - 3 + e - 2 \\ &= 5e\end{aligned}$$

Example 14.6
Find the value of e^2, rounded off to one decimal place.

Solution

We know

$$\begin{aligned}
e^2 &= 1+\frac{2}{1!}+\frac{2^2}{2!}+\frac{2^3}{3!}+\frac{2^4}{4!}+\frac{2^5}{5!}+\frac{2^6}{6!}+\dots\\
&= 1+2+2+\frac{4}{3}+\frac{2}{3}+\frac{4}{15}+\frac{4}{45}+\dots\\
&\geq \text{the sum of the first seven terms}\\
&\geq 7.355
\end{aligned}$$

On the other hand

$$\begin{aligned}
e^2 &\leq \left(1+\frac{2}{1!}+\frac{2^2}{2!}+\frac{2^3}{3!}+\frac{2^4}{4!}\right)+\frac{2^5}{5!}\left(1+\frac{2}{6}+\frac{2^2}{6^2}+\frac{2^3}{6^3}+\dots\right)\\
&= 7+\frac{4}{15}\frac{1}{1-\frac{1}{3}}\\
&= 7+\frac{2}{5}=7.4
\end{aligned}$$

Here, we have taken the first five terms of the series for e^2, and have compared the remaining terms with a suitable geometric series.

Thus we have proved that e^2 lies between 7.355 and 7.4. Therefore, the vlaue of e^2, rounded off to one decimal place is 7.4.

Remark

If we take 6 or more terms for exact calculation and compare the remaining with a suitable geometric series, we get a better approximation for the value of e^2.

EXERCISE 14.1

1. Find the value of e rounded off to one decimal place.
2. Find the coefficient of x^n in the expansion of e^{a+bx} in powers of x.
3. Find the sum of $1+\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+\dots$
4. Find the sum of
$$\sum_{n=1}^{\infty}\frac{n^2}{(n+1)!}$$
5. Prove that $\frac{e-e^{-1}}{2}=1+\frac{1}{3!}+\frac{1}{5!}+\dots$

6. Sum the series from $n = 1$ to ∞, whose nth term is

(i) $\frac{1}{(n+1)!}$

(ii) $\frac{1}{(n+2)!}$

(iii) $\frac{1}{(2n-1)!}$

(iv) $\frac{1}{(2n+1)!}$

7. Sum the following series:

(i) $1 + \frac{2^2}{2!} + \frac{3^2}{3!} + \frac{4^2}{4!} + \dots$

(ii) $1 + \frac{1+2}{2!} + \frac{1+2+3}{3!} + \frac{1+2+3+4}{4!} + \dots$

(iii) $\frac{2}{3!} + \frac{4}{5!} + \frac{6}{7!} + \frac{8}{9!} + \dots$

8. Sum the series:

(i) $\sum_{n=2}^{\infty} \frac{C(n,2)}{(n+1)!}$

(ii) $\sum_{n=1}^{\infty} \frac{C(n,0) + C(n,1) + \dots + C(n,n)}{P(n,n)}$

9. Write the series for

(i) $\frac{e^x - e^{-x}}{2}$

(ii) $e^{2x} + e^{-2x}$

10. Sum the series:

$$\sum_{n=1}^{\infty} \frac{2n}{n!}$$

14.2 Logarithmic Series

You have studied in earlier classes that $\log_a x$ is that number y such that $a^y = x$. Here a is known as the base of the logarithm. In this section we will take the number e as the base of the logarithm, whenever the base is not explicitly mentioned. Presently, we obtain an expansion for $\log(1+x)$ as a series of powers of x. This expansion will be valid only when $|x| < 1$.

Theorem 14.1

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots \text{ holds if } |x| < 1$$

[This is called the logarithmic series]

Proof: Consider the expansion of $(1+x)^y$ in different ways. First, we have the binomial series expansion:

$$(1+x)^y = 1 + y.x + \frac{y(y-1)}{1.2}.x^2 + \dots \qquad \text{[Here we use } |x| < 1\text{]}$$

Secondly, $(1+x)^y$ can be written as $e^{\log\{(1+x)^y\}}$ [This is because every number k may be written as $e^{\log k}$. This is how $\log k$ is defined.]

This is the same as $e^{y \log(1+x)}$ [Note that $1+x > 0$ and so $\log(1+x)$ is defined.]

[Here we use the result $\log(a^b) = b \log a$]

Now, this is the sum of the exponential series

$$1 + \frac{y \log(1+x)}{1!} + \frac{[y \log(1+x)]^2}{2!} + \dots$$

Thus we have two different series expansions for the same quantity $(1+x)^y$. In the latter series, the coefficient of y is $\log(1+x)$. In the former series the coefficient of y is

$$x - \frac{x^2}{1.2} + \frac{1.2}{1.2.3}x^3 - \frac{1.2.3}{1.2.3.4}x^4 + \dots$$

Explanation: In the term yx, the coefficient of y is x. In the term $\frac{y(y-1)}{1.2}x^2$, the coefficient of y is $\frac{-1}{1.2}x^2$. In the term $\frac{y(y-1)(y-2)}{1.2.3}x^3$, the coefficient of y is $\frac{1.2}{1.2.3}x^3$ and so on. Thus, equating the coefficients of y in the two expansions,

$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$

Note: In the above proof we have written $(1+x)^y$ as a series in powers of x and then we have collected coefficients of powers of y from the various terms. This step is not valid in general. But in this case, it can be shown that the step is valid.

Corollaries:

(i) $\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \dots$ if $|x| < 1$. (This is obtained from the theorem by substituting $-x$ wherever x occurs.)

Since all terms carry negative signs here, it is easier to remember it in the form

$-\log(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \dots$ if $|x| < 1$

(ii) Adding this series for $-\log(1-x)$ with the series for $\log(1+x)$, we obtain

$$\log(1+x)-\log(1-x) = \left[x-\frac{x^2}{2}+\frac{x^3}{3}-\dots\right]+\left[x+\frac{x^2}{2}+\frac{x^3}{3}+\dots\right]$$
$$= 2\left[x+\frac{x^3}{3}+\dots\right], \text{ if } |x|<1$$

The left side can also be written as

$$\log\frac{1+x}{1-x}=2\left[x+\frac{x^3}{3}+\dots\right] \text{ if } |x|<1.$$

Remark

The series expansion for $\log(1+x)$ may fail to be valid if $|x|$ is not less than 1. The series

$$\log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\dots$$

(i) When $x=2$, the series becomes $2-\frac{2^2}{2}+\frac{2^3}{3}-\dots$
Every term here is numerically greater than 1. We shall study in higher classes that such a series cannot have a sum.

(ii) When $x=-1$, the series does not have a sum. This is in conformity with the fact that $\log(1-1)$ is not a finite quantity.

(iii) When $x=1$, the series for $\log(1+x)$ becomes $\log 2=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\dots$
It can be proved that this is valid. We shall assume this validity and use it in the problems below.

Remark

Three major differences between the exponential series and the logarithmic series are:

1. In the series $e^x=1+\frac{x}{1!}+\frac{x^2}{2!}+\dots$, all terms carry positive signs. In the series $\log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\dots$, the terms carry alternately positive and negative signs.

2. In the logarithmic series, the factorial symbol does not occur. But in the exponential series, the denominators of the terms involve the factorials.

3. The exponential series is valid for all the values of x. The logarithmic series is valid when $|x|<1$.

Example 14.7
Find the sum of $\frac{5}{1.2.3}+\frac{7}{3.4.5}+\frac{9}{5.6.7}+\ldots$

Solution
The nth term here is

$$\frac{2n+3}{(2n-1)2n(2n+1)}$$

We first write this in the form
$\frac{A}{2n-1}+\frac{B}{2n}+\frac{C}{2n+1}$ where $A, B,$ and C are constants to be determined now. (This is called splitting into partial fractions).
Now,

$$\frac{A}{2n-1}+\frac{B}{2n}+\frac{C}{2n+1}=\frac{A(2n)(2n+1)+B(2n-1)(2n+1)+C(2n-1)2n}{(2n-1)2n(2n+1)}$$

We find A, B and C after equating

$$A(2n)(2n+1)+B(2n-1)(2n+1)+C(2n-1)2n=2n+3$$

We compare the coefficients of n^2, n and constant terms on the two sides of this equation:

$$4A+4B+4C = 0 \qquad (14.2)$$

$$2A-2C = 2 \qquad (14.3)$$

$$-B = 3 \qquad (14.4)$$

Solving this system of equations, we get $A=2$, $B=-3$ and $C=1$. Therefore, the nth term here is $\frac{2}{2n-1}-\frac{3}{2n}+\frac{1}{2n+1}$
Thus the given series is

$$\left(\frac{2}{1}-\frac{3}{2}+\frac{1}{3}\right)+\left(\frac{2}{3}-\frac{3}{4}+\frac{1}{5}\right)+\left(\frac{2}{5}-\frac{3}{6}+\frac{1}{7}\right)+\ldots$$

Now,

$$\begin{aligned}\frac{2}{1}-\frac{3}{2}+\frac{1}{3} &= \frac{2}{1}-\frac{2}{2}-\frac{1}{2}+\frac{1}{3}\\ &= 2\left[1-\frac{1}{2}\right]+\left[-\frac{1}{2}+\frac{1}{3}\right]\\ \left(\frac{2}{1}-\frac{3}{2}+\frac{1}{3}\right)+\left(\frac{2}{3}-\frac{3}{4}+\frac{1}{5}\right) &= 2\left[1-\frac{1}{2}\right]+\left[-\frac{1}{2}+\frac{1}{3}\right]+\left(\frac{2}{3}-\frac{2}{4}-\frac{1}{4}+\frac{1}{5}\right)\\ &= 2\left[1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}\right]+\left[-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}\right]\end{aligned}$$

So the series can be rewritten as

$$2\left[1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots\right]+\left[-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}\cdots\right]$$
$$= 2\log 2 + (\log 2) - 1$$
$$= 3\log 2 - 1$$

Example 14.8

Prove that $\log(n+1) - \log n = 2\left[\frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \frac{1}{5(2n+1)^5} + \ldots\right]$

Solution

$$\begin{aligned}
\text{Right hand side} &= 2\left[x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\right] \text{ where } x = \frac{1}{2n+1} \\
&= \log\frac{1+x}{1-x} \\
&= \log\frac{1+\frac{1}{2n+1}}{1-\frac{1}{2n+1}} \\
&= \log\frac{2n+2}{2n} \\
&= \log\frac{n+1}{n} \\
&= \log(n+1) - \log n = \text{Left hand side}
\end{aligned}$$

Example 14.9

If α, β are the roots of the equation $x^2 - px + q = 0$, prove that

$$\log(1+px+qx^2) = (\alpha+\beta)x - \frac{\alpha^2+\beta^2}{2}x^2 + \frac{\alpha^3+\beta^3}{3}x^3 - \ldots$$

Solution

$$\begin{aligned}
\text{Right hand side} &= \left[\alpha x - \frac{\alpha^2x^2}{2} + \frac{\alpha^3x^3}{3} - \ldots\right] + \left[\beta x - \frac{\beta^2x^2}{2} + \frac{\beta^3x^3}{3} - \ldots\right] \\
&= \log(1+\alpha x) + \log(1+\beta x) \\
&= \log\left[(1+\alpha x)\times(1+\beta x)\right] \\
&= \log\left[1+(\alpha+\beta)x+\alpha\beta x^2\right] \\
&= \log(1+px+qx^2) \\
&= \text{Left hand side.}
\end{aligned}$$

Here we have used the facts $\alpha + \beta = p$ and $\alpha\beta = q$. We know this from the chapter on quadratic equations. We have also assumed that both $|\alpha x|$ and $|\beta x|$ are <1.

Example 14.10
Using the series for $\log 2$, prove that the value of $\log 2$ lies between 0.61 and 0.76.

Solution

$$\begin{aligned}
\log 2 = \log(1+1) &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots \\
&= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) + \dots \\
&= \frac{1}{2} + \frac{1}{12} + \frac{1}{30} + \dots \\
&\geq \frac{37}{60} > 0.616 \\
\text{Also } \log 2 &= 1 - \left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{1}{4} - \frac{1}{5}\right) - \left(\frac{1}{6} - \frac{1}{7}\right) - \dots \\
&= 1 - \frac{1}{6} - \frac{1}{20} - \frac{1}{42} - \dots \\
&\leq 1 - \frac{1}{6} - \frac{1}{20} - \frac{1}{42} \\
&= \frac{319}{420} \\
&< 0.76
\end{aligned}$$

Hence the result.

Example 14.11
Using a suitable logarithmic series, find an approximate value of $\log 3$.

Solution
An approximate value of e is 2.7

$\therefore$ $\log 3$ is approximately

$$\begin{aligned}
\log \frac{3.0e}{2.7} &= \log \frac{30e}{27} \\
&= \log e + \log \frac{10}{9} \\
&= 1 + \log\left(1 + \frac{1}{9}\right)
\end{aligned}$$

$$= 1 + \frac{1}{9} - \frac{1}{2}\frac{1}{81} + \frac{1}{3}\frac{1}{729} - \cdots = 1.105 \quad \text{approximately.}$$

EXERCISE 14.2

1. Prove that $\log 2 < 1 < \log 3$. [Hint: Convert this to a problem regarding e].
2. If $y = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$ and if $|x| < 1$, prove that $x = y + \frac{y^2}{2!} + \frac{y^3}{3!} + \ldots$
3. Prove that the series $\frac{1}{n+1} + \frac{1}{2(n+1)^2} + \frac{1}{3(n+1)^3} + \ldots$ has the same sum as the series $\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \ldots$
4. Prove that $\frac{1}{1.2.3} + \frac{1}{3.4.5} + \frac{1}{5.6.7} + \ldots = \log 2 - \frac{1}{2}$.
5. Prove that $\log(1+x)^{1+x}(1-x)^{1-x} = 2\left[\frac{x^2}{1.2} + \frac{x^4}{3.4} + \frac{x^6}{5.6} + \ldots\right]$
6. Prove that $2\log x - \log(x+1) - \log(x-1) = \frac{1}{x^2} + \frac{1}{2x^4} + \frac{1}{3x^6} + \ldots$
7. Find the value of $\log 4$ correct to one decimal place.

MISCELLANEOUS EXERCISE ON CHAPTER 14

1. Evaluate:

$$(x-y) + \frac{1}{2!}(x^2 - y^2) + \frac{1}{3!}(x^3 - y^3) + \frac{1}{4!}(x^4 - y^4) + \ldots$$

2. Find the value of $e^{-\frac{1}{4}}$ upto 4 places of decimals.
3. Show that

$$\log \frac{1+3x}{e^{1-2x}} = 5x - \frac{5}{2}x^2 + \frac{35}{3}x^3 - \frac{65}{4}x^4 + \ldots$$

4. Prove that

$$\log(1 + 3x + 2x^2) = 3x - \frac{5}{2}x^2 + \frac{9}{3}x^3 - \frac{17}{4}x^4 + \ldots$$

CHAPTER 15

Solution of Triangles

While introducing trigonometry in Chapter 2, we had stated that a principal objective of trigonometry was to be able to calculate sides and angles of a triangle when some of its sides and angles are given.

15.1 Some Basic Formulas

Let the angles of a triangle be denoted by A, B and C and the sides opposite to them by a, b and c respectively.

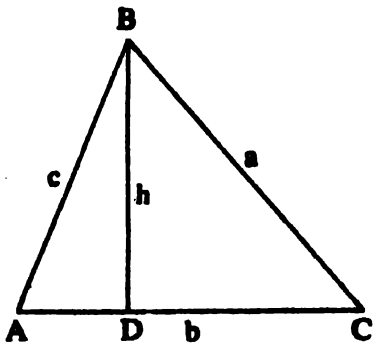


Fig.15.1

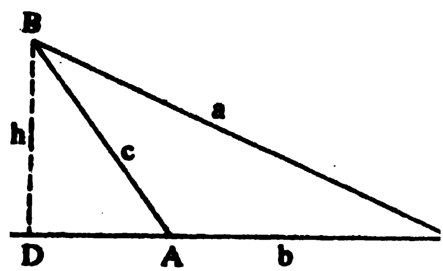


Fig 15.2

From Figs 15.1 and 15.2, we see that

$$\frac{h}{c} = \sin A \text{ and } \frac{h}{a} = \sin C$$

These two relations hold whether the angle A is acute or obtuse. It obviously holds if A is a right angle. Thus $c \sin A = a \sin C$ or $\frac{\sin A}{a} = \frac{\sin C}{c}$. Similarly, drawing the altitude from the vertex C, we can show that

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

Therefore, we have

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

These equations constitute the *Law of Sines*. From this we see that in any triangle, the length of the sides are proportional to the sines of the opposite angles.

The Law of Cosines

Referring back to Fig 15.1, we have

$$\begin{aligned} AB^2 &= BD^2 + AD^2 \\ &= BD^2 + (AC - CD)^2 \\ &= BD^2 + AC^2 + CD^2 - 2AC.CD \end{aligned}$$

Now, $BD^2 + CD^2 = BC^2$ and $CD = BC\cos C$. Hence,

$$\begin{aligned} AB^2 &= AC^2 + BC^2 - 2AC.BC\cos C \\ \text{or} \qquad c^2 &= a^2 + b^2 - 2ab\cos C \end{aligned} \tag{15.1}$$

The same equation is obtained if the angle A is obtuse as in Fig. 15.2. We also have two other equations obtained in a similar way.

$$b^2 = a^2 + c^2 - 2ac\cos B \tag{15.2}$$

$$a^2 = b^2 + c^2 - 2bc\cos A \tag{15.3}$$

The equations (15.1), (15.2), and (15.3) constitute the law of cosines. A convenient form of the cosine law, when the angles are to be found, is

$$\begin{aligned} \cos A &= \frac{b^2 + c^2 - a^2}{2bc} \\ \cos B &= \frac{c^2 + a^2 - b^2}{2ca} \\ \cos C &= \frac{a^2 + b^2 - c^2}{2ab} \end{aligned}$$

Half Angles

We have

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

Hence,

$$2\sin^2\frac{A}{2} = 1 - \cos A$$

$$= \frac{a^2 - (b-c)^2}{2bc}$$

$$= \frac{(a+b-c)(a-b+c)}{2bc}$$

Let $\qquad a + b + c = 2s.$

Then $\qquad \sin^2 \frac{A}{2} = \frac{4(s-b)(s-c)}{4bc}$

or $\qquad \sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}} \qquad (15.4)$

Similarly, we obtain

$$\sin \frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{ca}} \qquad (15.5)$$

and $\qquad \sin \frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}} \qquad (15.6)$

Also,

$$2\cos^2 \frac{A}{2} = 1 + \cos A$$

$$= \frac{(b+c)^2 - a^2}{2bc} = \frac{(b+c+a)(b+c-a)}{2bc}$$

Hence, as before

$$\cos \frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}. \qquad (15.7)$$

$$\cos \frac{B}{2} = \sqrt{\frac{s(s-b)}{ca}} \qquad (15.8)$$

$$\cos \frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}} \qquad (15.9)$$

Using formulas (15.4) to (15.9), we have

$$\tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} \qquad (15.10)$$

$$\tan \frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}} \qquad (15.11)$$

$$\tan \frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}} \qquad (15.12)$$

The Area of a Triangle

Referring again to Figs 15.1 and 15.2, we see that the area Δ of the triangle ABC is given by

$$\Delta = \frac{1}{2}bc\sin A.$$

Similarly, we can show that $\Delta = \frac{1}{2}ca\sin B = \frac{1}{2}ab\sin C$. Now,

$$\begin{aligned}\Delta &= \frac{1}{2}bc\sin A = bc\sin\frac{A}{2}\cos\frac{A}{2}\\ &= bc\sqrt{\frac{(s-b)(s-c)}{bc}}\sqrt{\frac{s(s-a)}{bc}} \quad \text{or} \quad \Delta = \sqrt{s(s-a)(s-b)(s-c)}\end{aligned}$$

This formula is known as Hero's formula.

Example 15.1
Given $A = 25$, $b = 52$, $c = 63$. Find $\cos A$.

Solution

$$\begin{aligned}\cos A &= \frac{b^2+c^2-a^2}{2bc}\\ &= \frac{52^2+63^2-25^2}{2\times 52\times 63} = \frac{2704+3969-625}{2\times 52\times 63}\\ &= \frac{12}{13}.\end{aligned}$$

Hence, $\cos A = \frac{12}{13}$.

Example 15.2
Given $a = 15$, $b = 36$, $c = 39$. Find $\sin\frac{A}{2}$.

Solution

$$\begin{aligned}\sin\frac{A}{2} &= \sqrt{\frac{(s-b)(s-c)}{bc}}; 2s = 90\\ &= \sqrt{\frac{(45-36)(45-39)}{36\times 39}}\\ &= \sqrt{\frac{9\times 6}{36\times 39}} = \sqrt{\frac{1}{26}}\end{aligned}$$

EXERCISE 15.1

1. Given a $= 18$, b $= 24$, c $= 30$. Find

   (i) $\sin A$, $\sin B$, $\sin C$

   (ii) $\tan A$, $\tan B$, $\tan C$

   (iii) Δ

   (iv) $\tan \frac{A}{2}$, $\tan \frac{B}{2}$, $\tan \frac{C}{2}$.

15.2 Some More Formulas

1. In any triangle ABC, we have

$$\begin{aligned} a &= b\cos C + c\cos B \\ b &= c\cos A + a\cos C \\ c &= a\cos B + b\cos A. \end{aligned}$$

Proof: Referring again to Fig. 15.1, we have

$$b = AD + DC, AD = c\cos A \text{ and } DC = a\cos C.$$

Hence $b = c\cos A + a\cos C$.

If A is obtuse (See Fig. 15.2), we have

$$b = CD - AD, CD = a\cos C, \quad \begin{aligned} AD &= c\cos(\pi - A) \\ &= -c\cos A \end{aligned}$$

Hence, $b = a\cos C - (-c\cos A) = a\cos C + c\cos A$.

Other formulas are proved in similar manner.

2. *Law of tangents*: In a triangle ABC,

$$\tan\frac{B-C}{2} = \frac{b-c}{b+c}\cot\frac{A}{2}$$

Proof: By law of sines, we have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k, \quad \text{say.}$$

Therefore,

$$\frac{b-c}{b+c} = \frac{k(\sin B - \sin C)}{k(\sin B + \sin C)} = \frac{\sin B - \sin C}{\sin B + \sin C}$$
$$= \frac{2\cos\frac{B+C}{2}\sin\frac{B-C}{2}}{2\sin\frac{B+C}{2}\cos\frac{B-C}{2}}$$
$$= \tan\frac{B-C}{2}\quad\frac{\cos\left(\frac{\pi}{2}-\frac{A}{2}\right)}{\sin\left(\frac{\pi}{2}-\frac{A}{2}\right)} = \tan\frac{B-C}{2}\quad\frac{\sin\frac{A}{2}}{\cos\frac{A}{2}}$$

or $$\tan\frac{B-C}{2} = \frac{b-c}{b+c}\cot\frac{A}{2}.$$

We have similar formulas for $\tan\frac{C-A}{2}$ and $\tan\frac{A-B}{2}$. The above formula can also be written as

$$\frac{b-c}{b+c} = \frac{\tan\frac{B-C}{2}}{\tan\frac{B+C}{2}}$$

The student is advised to prove it.

EXERCISE 15.2

1. Write formulas for $\tan\frac{C-A}{2}$ and $\tan\frac{A-B}{2}$ analogous to that of $\tan\frac{B-C}{2}$ and prove them.
2. Prove that

$$\frac{a+b}{c} = \frac{\cos\frac{A-B}{2}}{\sin\frac{C}{2}},$$
$$\frac{a-b}{c} = \frac{\sin\frac{A-B}{2}}{\cos\frac{C}{2}}$$

Half Angle Formulas

ABC is a triangle. Let AO, BO and CO be angle bisectors. Then $OD = OE = OF = r$, where r is the radius of the incircle. From Fig. 15.3, we see that $\tan\frac{A}{2} = \frac{r}{AF}$,

$\tan\frac{B}{2} = \frac{r}{BD}$ and $\tan\frac{C}{2} = \frac{r}{CE}$. We shall find expressions for r, AF, BD and CE in terms of the sides a, b and c. The area Δ of the triangle ABC is the sum of the areas of the triangles AOB, BOC and AOC. Hence,

$$\Delta = \frac{1}{2}cr + \frac{1}{2}ar + \frac{1}{2}br = \frac{1}{2}(a+b+c)r = rs$$

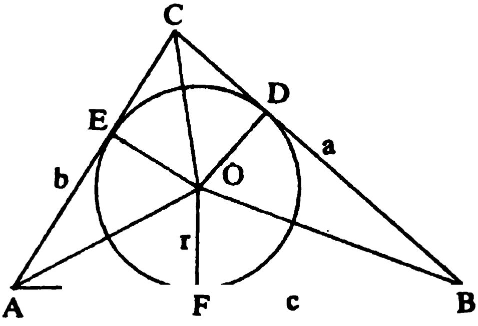


Fig 15.3

Therefore, $r = \frac{\Delta}{s}$.
Also triangle AFO and AEO, are congruent. Hence, $AF = AE$. Similarly, $BF = BD$ and $CE = CD$. Since the sum of these six segments is $a+b+c = 2s$, we get

$$AF + BD + CD = s.$$

Therefore $AF = s - (BD + CD) = s - a$. Similarly, $BD = s - b$ and $CE = s - c$. Hence,

$$\tan\frac{A}{2} = \frac{r}{s-a}, \tan\frac{B}{2} = \frac{r}{s-b} \quad \text{and} \quad \tan\frac{C}{2} = \frac{r}{s-c}.$$

These formulas give a quick way of finding the angles of a triangle, given its sides.

Example 15.3
If $a = 13$, $b = 14$, $c = 15$, find $\tan\frac{A}{2}$, $\tan\frac{B}{2}$, $\tan\frac{C}{2}$, r and Δ.

Solution

$$\begin{aligned} s &= \frac{13+14+15}{2} = 21 \\ \Delta &= \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{21 \times 8 \times 7 \times 6} \\ &= \sqrt{7 \times 7 \times 6 \times 6 \times 4} = 84 \text{ sq.unit} \\ r &= \frac{\Delta}{s} = 4; \ \tan\frac{A}{2} = \frac{4}{21-13} = \frac{1}{2} \\ \tan\frac{B}{2} &= \frac{4}{7}, \tan\frac{C}{2} = \frac{4}{6} = \frac{2}{3} \end{aligned}$$

Example 15.4
If the sides a, b, c of a triangle are in Arithmetic Progression, prove that

$$\cot\frac{A}{2}, \cot\frac{B}{2}, \cot\frac{C}{2} \quad \text{are in} \quad A.P.$$

Solution

To show that $\cot\frac{A}{2} + \cot\frac{C}{2} = 2\cot\frac{B}{2}$.

$$\begin{aligned} L.H.S. &= \cot\frac{A}{2} + \cot\frac{C}{2} \\ &= \frac{s-a}{r} + \frac{s-c}{r} = \frac{2s-(a+c)}{r} \\ &= \frac{2s-2b}{r} = 2(\frac{s-b}{r}) = 2\cot\frac{B}{2} \end{aligned}$$

[Since a, b, c are in A.P.

$\therefore 2b = a + c$]

Example 15.5

Let R be the radius of the circumcircle of a triangle ABC. Show that

$$R = \frac{a}{2\sin A} = \frac{b}{2\sin B} = \frac{c}{2\sin C}$$

Solution

Let O be the circumcentre and OD be the perpendicular on BC from O.Then $BD = CD = \frac{a}{2}$. Also triangles OBD and OCD are congruent. Furthermore, $\angle BOC = 2A$. Hence, $\angle BOD = A$. Therefore, $\sin A = \frac{BD}{BO} = \frac{\frac{a}{2}}{R}$. Hence, $R = \frac{a}{2\sin A}$. Hence, by law of sines $R = \frac{a}{2\sin A} = \frac{b}{2\sin B} = \frac{c}{2\sin C}$

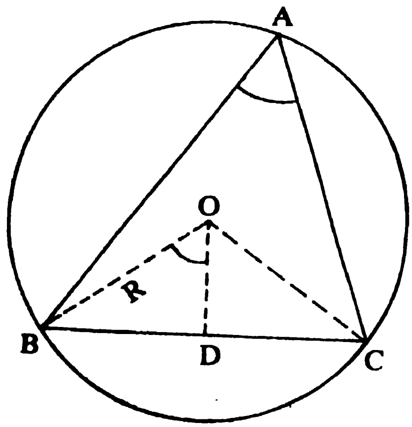


Fig 15.4

EXERCISE 15.3

In any triangle ABC, prove that

1. $\sin\frac{B-C}{2} = \frac{b-c}{a}\cos\frac{A}{2}$
2. $a(b\cos C - c\cos B) = b^2 - c^2$
3. $a(\cos C - \cos B) = 2(b-c)\cos^2\frac{A}{2}$
4. $\frac{\sin(B-C)}{\sin(B+C)} = \frac{b^2-c^2}{a^2}$

5. $a\sin(B-C)+b\sin(C-A)+c\sin(A-B)=0$

6. Find the angles A, B, C and also the radius of incircle, given

   (i) $a=18,\ b=24,\ c=30$

   (ii) $a=13,\ b=4,\ c=15$

   Use trigonometric tables, if necessary.

7. Let R be the radius of the circumcircle of a triangle ABC. Show that $R=\frac{abc}{4\Delta}$.

8. The sides of a triangle are 22, 28, 36 centimetres. Find the areas of the inscribed circle, the triangle and the circumscribed circle.

15.3 Right Triangles

Let us consider the problem of solving right triangles. In general a triangle can be solved if we know three of its parts, at least one of which is a side. Thus it is possible to find the remaining parts of a right triangle if in addition to the right angle, one side and any other part (side or angle) are known. Thus if $A=90°$, side b and angle B are given, then

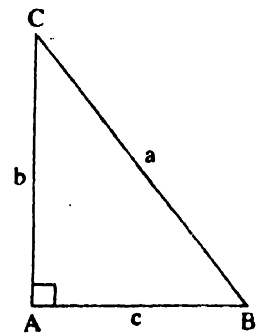


Fig 15.5

$$\frac{b}{\sin B}=\frac{a}{\sin 90°}$$

Hence $a=\frac{b}{\sin B}$. Also $\frac{c}{b}=\cot B$, so that $c=b\cot B$. If the sides b and c are given then $a=\sqrt{b^2+c^2}$. Similarly, we can solve the triangle in the other cases.

Example 15.6

A right triangle has $c=64, \angle A=61°$ and $\angle C=90°$. Find the ramaining parts.

Solution

Since $\angle A+\angle B+\angle C=180°$, we have $\angle B=180°-(61°+90°)=29°$.

Now $\frac{b}{\sin B}=\frac{c}{\sin 90°}$. Hence

$$b=c\sin B=64\sin 29° \quad \text{or} \quad b=64\times .4848=31.0272 \quad \text{(from tables)}.$$

Also, $a=64\cos 29°=64\times .8746=55.9744$

Example 15.7
Solve the right triangle ABC, given $a = 45, b = 35.2$ and $\angle C = 90°$.

Solution

$$\tan B = \frac{b}{a} = \frac{35.2}{45} = .7822$$

From tables, $B = 38°2'$ approximately.
Therefore $A = 51°58'$.
Also

$$\begin{aligned}\frac{c}{a} &= \sec B \text{ or } c = 45 \times 1.2696\\ &= 57.13\end{aligned}$$

EXERCISE 15.4

Solve the right triangles, where $\angle C = 90°$

1. $a = 43, A = 53°$
2. $c = 6.5, A = 26°$
3. $a = 50.4, b = 26.2$
4. $b = 3.3, c = 4.4$
5. $b = 4.5, A = 39°$
6. $a = 412, c = 610$

15.4 Oblique Triangles

We now consider the problem of solving oblique triangles – the tirangles which are not right angled. As already stated, we can solve a triangle if we know three of its parts at least one of which is a side. Different cases to be considered are:

Case I: The three sides are given.
Case II: Two sides and the included angle are given.
Case III: Two sides and the angle opposite to one of them are given.
Case IV: One side and two angles are given.

Case I: Given the three sides a, b and c

Since a, b, c are given, $s = \frac{a+b+c}{2}$ is known. Then the half angles can be computed by using formulas for their sines, cosines or tangents. Only two angles need be found, the third can be found by subtracting the sum of the two from 180°. The angles can also be found by using cosine formulas. These are usually avoided unless a, b and c are small numbers, as it may involve longer calculations.

Example 15.8

Solve the triangle for given $a = 20$, $b = 30$ and $c = 21$

Solution

$$\begin{aligned}
s &= \frac{20+30+21}{2} = 35.5 \\
\cos\frac{A}{2} &= \sqrt{\frac{s(s-a)}{b\times c}} = \sqrt{\frac{35.5(35.5-20)}{30\times 21}} \\
\log\cos\frac{A}{2} &= \frac{1}{2}[\log 35.5 + \log 15.5 - \log 30 - \log 21] \\
&= \frac{1}{2}[1.5502 + 1.1903 - 1.4771 - 1.3222] \\
&= \frac{1}{2}[2.7405 - 2.7993] \\
&= \frac{1}{2}[-.0588] = -.0294
\end{aligned}$$

Logarithmic cosine is defined by $L\cos\frac{A}{2} = 10 + \log\cos\frac{A}{2}$ (Other logarithmic trigonometric functions are defined similarly and there are tables which give their values).

Hence, $L\cos\frac{A}{2} = 10 - .0294 = 9.9706$.

From tables $\frac{A}{2} = 20°50'$

Hence $A = 41°40'$

Now $\quad \cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ca}} = \sqrt{\frac{35.5\times 5.5}{20\times 21}}$

$$\begin{aligned}
\log\cos\frac{B}{2} &= \frac{1}{2}[\log 35.5 + \log 5.5 - \log 20 - \log 21] \\
&= \frac{1}{2}[1\ 5502 + 0.7404 - 1.3010 - 1.3222] \\
&= -\frac{1}{2}[.3326] = -.1663
\end{aligned}$$

$L\cos\frac{B}{2} = 9.8337$.

Hence, $\frac{B}{2} = 47°$. so that $B = 94°$.

Therefore, $C = 180° - (A + B) = 44°20'$.

EXERCISE 15.5

1. If $a = 9$, $b = 10$, $c = 11$, find B, given $\log 2 = .30103$, $L \tan 29°29' = 9.7523472$ and $L \tan 29°30' = 9.7526420$.

   [Hint: Use formula for $\tan \frac{B}{2}$ and interpolate.]

2. Solve the triangle, given
$$a = 2, b = \sqrt{6}, c = \sqrt{3} - 1$$
   [Hint: Use cosine formula.]

3. The sides of a triangle are 2, 3 and 4. Find the greatest angle, if $\log 2 = .30103$, $\log 3 = .4771213$, $L \tan 52°14' = 10.1108395$ and $L \tan 52°15' = 10.1111004$

4. Making use of the tables, solve the triangle, given $a = 25, b = 26$ and $c = 27$

Case II: *Given two sides b and c and the included angle A*

Let b denote the greater side. Then we have
$$\tan \frac{B-C}{2} = \frac{b-c}{b+c} \cot \frac{A}{2}$$
This gives us the angle $\frac{B-C}{2}$. Since, $\frac{B+C}{2} = 90° - \frac{A}{2}$ we now get the angles B and C. The third side can now be determined by Sine law. The third side can also be found by Cosine law, but as mentioned earlier, we avoid it unless the sides are small.

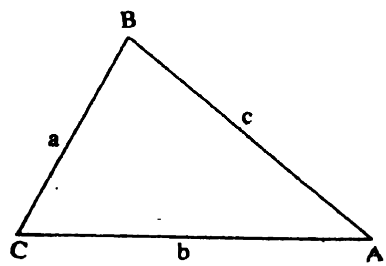


Fig 15.6

Example 15.9
Solve the triangle, given
$$b = 50, \quad c = 80, \quad A = 132°$$

Solution

$$\tan \frac{C-B}{2} = \frac{c-b}{c+b} \cot \frac{A}{2} = \frac{30}{130} \cot 66°$$

Hence

$$\begin{aligned} L\tan\frac{C-B}{2} &= \log 3 + L\cot 66^\circ - \log 13 \\ &= .4771 + 9.6486 - 1.1139 \\ &= 10.1257 - 1.1139 = 9.0118 \end{aligned}$$

Hence $\frac{C-B}{2} = 5^\circ 50'$ or $C - B = 11^\circ 40'$.
Also $\frac{C+B}{2} = 90^\circ - 66^\circ = 24^\circ$ or $C + B = 48^\circ$.
Therefore, $C = 29^\circ 50'$, $B = 18^\circ 10'$
Now $\frac{a}{\sin A} = \frac{c}{\sin C}$, Hence

$$a = \frac{c\sin A}{\sin C} = \frac{80\sin 132^\circ}{\sin 29^\circ 50'}$$

Hence,

$$\begin{aligned} \log a &= \log 80 + L\sin 132^\circ - L\sin 29^\circ 50' \\ &= 1.9031 + L\cos 42^\circ - L\sin 29^\circ 50' \end{aligned}$$

or

$$\begin{aligned} \log a &= 1.9031 + 9.8711 - 9.6968 \\ &= 11.7742 - 9.6968 \\ &= 2.0774 \end{aligned}$$

Hence, $a = 119.5$ (from tables).

EXERCISE 15.6

1. If $a = 21$, $b = 11$ and $C = 34^\circ 42' 30''$, find A and B, given $\log 2 = .30103$, $L\tan 72^\circ 38' 45'' = 10.50515$.
2. If $b = 14$, $c = 11$ and $A = 60^\circ$, find B and C, given $\log 2 = .30103$, $\log 3 = .4771213$ $L\tan 11^\circ 44' = 9.3174299$, $L\tan 11^\circ 45' = 9.3180640$
3. If $a = \sqrt{3} + 1$, $b = \sqrt{3} - 1$ and $C = 60^\circ$, find the other side and angles.
4. $a = 40$, $c = 40\sqrt{3}$ and $B = 30^\circ$, solve the triangle.

Case III: Given two sides b and c and the angle B opposite to one of them.

In this case there may be no triangle, or one triangle or two triangles depending on the given parts. For this reason, it is called the *ambiguous case*. These may be considered as follows:
By cosine law

$$\begin{aligned} b^2 &= c^2 + a^2 - 2ca\cos B \\ \text{or}\quad a^2 - 2ac\cos B &= b^2 - c^2. \end{aligned}$$

Adding $c^2\cos^2 B$ to both sides, we get

$$\begin{aligned} &a^2 - 2ac\cos B + c^2\cos^2 B \\ &= b^2 - c^2 + c^2\cos^2 B \\ &= b^2 - c^2\sin^2 B. \end{aligned}$$

Thus, $$(a - c\cos B)^2 = b^2 - c^2\sin^2 B$$

or $$a = c\cos B \pm \sqrt{b^2 - c^2\sin^2 B}.$$

This equation helps us to determine a, given b, c and B.

(i) If $b < c\sin B$, then $b^2 - c^2\sin^2 B < 0$ and hence there is no solution and hence no triangle.

(ii) If $b = c\sin B$, then $a = c\cos B$ (only one solution) and the triangle is right angled. Note that $\frac{b}{\sin B} = c$. Hence $\angle C = 90^\circ$.

(iii) If $b > c\sin B$, then there are two solutions. But, since $a > 0$, the value $c\cos B - \sqrt{b^2 - c^2\sin^2 B}$ is inadmissible unless it is positive, i.e. unless

$$\sqrt{b^2 - c^2\sin^2 B} < c\cos B$$

or $$b^2 - c^2\sin^2 B < c^2\cos^2 B$$

or $$b^2 < c^2.$$

Hence there are two triangles when $c > b > c\sin B$

(iv) If B is obtuse, then $c\cos B$ is negative, and one value of a is negative and the corresponding triangle is impossible. The other value will be positive only when

$$c\cos B + \sqrt{b^2 - c^2\sin^2 B} > 0$$

or $$\sqrt{b^2 - c^2\sin^2 B} > -c\cos B$$

or $$b^2 - c^2\sin^2 B > c^2\cos^2 B.$$

or $$b^2 > c^2 \quad \text{or} \quad b > c.$$

Hence, if B is obtuse, there is only one triangle when $b > c$.

Example 15.10
Given $a = 2528$, $b = 3126$, $B = 51°25'$, solve the triangle.

Solution
In the discussion, we had used the letters b, c and B for the given parts. When other letters are used (as in this case) the reasoning remains the same. In this case the side b (opposite to angle B) $> a$ and form (iii), we have exactly one triangle.

Now, $\frac{a}{\sin A} = \frac{b}{\sin B}$
Hence, $\sin A = \frac{a}{b}\sin B$.
Therefore,

$$\begin{aligned} L\sin A &= \log 2528 + L\sin 51^\circ 25' - \log 3126 \\ &= 3.4028 + 9.8930 - 3.4950 \quad \text{(from tables)} \\ &= 13.2958 - 3.4950 \\ &= 9.8008 \end{aligned}$$

Hence, $A = 39^\circ 13'$. Therefore, $C = 180^\circ - (51^\circ 25' + 39^\circ 13')$

$$= 89^\circ 22'$$

Also $c = \frac{b\sin C}{\sin B}$
Hence, $\log c = \log 3126 + L\sin 89^\circ 22' - L\sin 51^\circ 25'$

$$\begin{aligned} &= 3.4950 + 10.0000 - 9.8930 \\ &= 3.6020 \end{aligned}$$

Hence, $c = 3999$.

Example 15.11
Solve the triangle for $b = 50$, $c = 63$ and $B = 54^\circ$.

Solution
Let us calculate $c\sin B$ to see whether $b < c\sin B$ or not. In fact we calculate

$$\frac{c\sin B}{b} = \sin C.$$

Now

$$\begin{aligned} \log\sin C &= \log 63 + L\sin 54^\circ - 10 - \log 50 \\ &= 1.7993 + 9.9080 - 10 - 1.6990 \\ &= .0083 \end{aligned}$$

Since $\log\sin C > 0$, we conclude $\sin C > 1$. This implies that there is no solution.

Example 15.12
Solve the triangle for $a = 28$, $b = 38$ and $A = 39^\circ$.

Solution

Here $a < b$ and $b \sin A = 38 \times .6293 < a$.
By (iii) there are two solutions.

Now,
$$c = b \cos A \pm \sqrt{a^2 - b^2 \sin^2 A}$$
$$\begin{aligned}\log(b \cos A) &= \log 38 + L \cos A - 10 = 1.5798 + 9.8905 - 10\\ &= 11.4703 - 10 = 1.4703\end{aligned}$$

Hence, $b \cos A = 29.53$
Similarly,

$$\begin{aligned}\log(b \sin A) &= 1.5798 + 9.7989 - 10\\ &= 11.3787 - 10 = 1.3787\end{aligned}$$

and $b \sin A = 23.91$. Also $2 \log(b \sin A) = 2.7574$.

Therefore, $b^2 \sin^2 A = 572$

Hence, $a^2 - b^2 \sin^2 A = 784 - 572 = 212.0$

Hence, $\sqrt{a^2 - b^2 \sin^2 A} = 14.56$.

Hence,
$$\begin{aligned}c &= 29.53 \pm (14.56)\\ &= 44.09 \quad \text{or} \quad 14.97\end{aligned}$$

Now,
$$\frac{\sin B}{b} = \frac{\sin A}{a}$$

Hence, $\sin B = \frac{38}{28} \sin 39°$.
Hence,

$$\begin{aligned}L \sin B &= 1.5798 + 9.7989 - 1.4472\\ &= 11.3787 - 1.4472\\ &= 9.9315\end{aligned}$$

Hence, $B = 58°40'$ or $180° - 58°40'$ i.e. $121°20'$

and
$$\begin{aligned}C &= 180° - (A + B)\\ &= 82°20' \quad \text{or} \quad 19°40'\end{aligned}$$

So the solutions are ($\triangle ABC$ in Fig. 15.7)

$$\begin{array}{lll} a = 28 & b = 38 & c = 44.09\\ A = 39° & B = 58°40' & C = 82°20'\end{array}$$

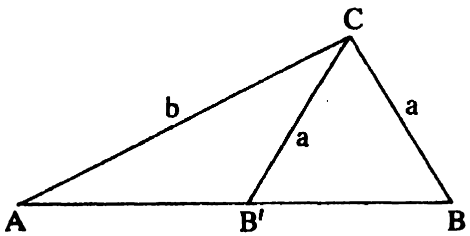


Fig. 15.7

For the other solution ($\triangle AB'C$ in Fig. 15.7),

$$B' = 121°20'$$
$$C = 19°40'$$
$$c = 15.40$$

EXERCISE 15.7

1. If $a = 5$, $b = 7$ and $\sin A = \frac{3}{4}$, solve the triangle, if possible.
2. If $A = 30°$, $a = 100$, $c = 100\sqrt{2}$, solve the triangle.
3. If $A = 30°$, $b = 8$ and $a = 6$, find c.
4. If a, b and A are given so that two triangles may be formed, show that the angles B and B' in the two triangles are supplementary to each other.
5. If $a = 5$, $b = 4$ and $A = 45°$, find the other angles, having given $\log 2 = .30103$, $L\sin 34°26' = 9.7523919$ and $L\sin 34°27' = 9.7525761$.

Case IV: Given one side and two angles, say, a, B and C

If two angles of a triangle are given then third can be found easily.
Thus $A = 180° - (B + C)$
The other sides can be found out by using sine formula. Thus

$$b = a\frac{\sin B}{\sin A} \quad \text{and} \quad c = a\frac{\sin C}{\sin A}$$

Example 15.13

Solve the triangle given

$$c = 1.732, \quad A = 23°16', \quad B = 20°3'.$$

Solution

$$\begin{aligned} C &= 180° - (23°16' + 20°3') \\ &= 136°41' \\ a &= \frac{c}{\sin C} \sin A. \end{aligned}$$

Hence,

$$\begin{aligned} \log a &= .2385 + 9.5966 - 9.8363 \\ &= -.0012 = \bar{1}.9988. \end{aligned}$$

Hence, $a = .9973$. The side b can be found similarly.

EXERCISE 15.8

1. Solve the triangles given

   (i) $c = 72, A = 56°, B = 65°$

   (ii) $a = 18, B = 108°, A = 25°$.

   (iii) $b = 302, A = 50°10', C = 72°$.

2. The base angles of a triangle are $22\frac{1}{2}^{\circ}$ and $112\frac{1}{2}^{\circ}$. Show that the base is equal to twice the height.

3. The angles of a triangle are in the ratio of 1:2:7. Show that the ratio of the greatest side to the least side is $\sqrt{5} + 1 : \sqrt{5} - 1$.

15.5 Heights and Distances

In this section we shall discuss some problems regarding heights and distances. We begin by defining some terms which are needed in our discussion.

Angles of Elevation and Angles of Depression

Suppose O and P are two points, P being at a higher level than O. Let OM be the horizontal line drawn through O to meet the vertical line drawn through P at M. The angle MOP is called the *angle of elevation* of the point P as seen from O. Draw PN, parallel to OM, so that PN is the horizontal line passing through P. The angle NPO is called the *angle of depression* of the point O as seen from P.

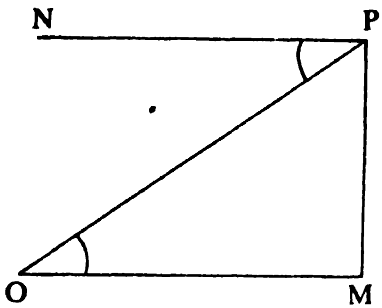


Fig 15.8

Bearings of a Point

Let NS stand for a line in the north-south direction and EW in the east-west direction. The acute angle which OA makes with NS is called the bearing of the point A from O. The bearing of A may be precisely indicated by giving the size of the angle and specifying whether it is measured from ON or OS and whether to the east or west. Thus in the figure, OA is in the direction 30° east of north and the bearing is written $N30°E$. The bearings of some other points are also indicated in Fig 15.9.

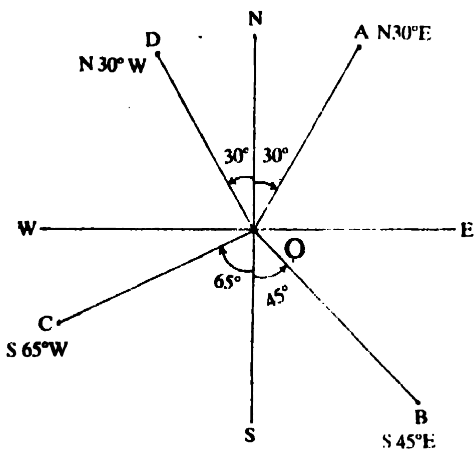


Fig 15.9

Example on Heights and Distances

Example 15.14

From a cliff 150 metre above the shore line the angle of depression of a ship is 19°30′. Find the distance from the ship to a point on the shore directly below the observer.

Solution

We want to find x in Fig. 15.10.

$$\text{Now,}\quad \frac{150}{x} = \tan 19°30'$$
$$= .3541$$

Hence, $x = \dfrac{150}{.3541} m$
$= 424\text{m}$ approximately.

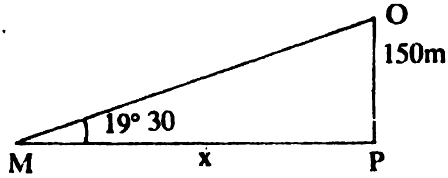


Fig 15.10

Example 15.15

From a certain point on a level street, the angle of elevation of the top of a building is 50°. From another point 100 metres further away from this point, the angle of elevation is 38°. Find the height of the building.

Solution

We want to find x in Fig.15.11,

$\dfrac{h}{\sin 38^\circ} = \dfrac{100}{\sin 12^\circ}$ and $x = h \sin 50^\circ$

Hence, $x = \dfrac{100 \sin 38^\circ \sin 50^\circ}{\sin 12^\circ}$

$$\begin{aligned} \text{Now } \log x &= 2 + L\sin 38^\circ + L\sin 50^\circ - 10 - L\sin 12^\circ \\ &= 2 + 9.7893 + 9.8843 - 10 - 9.3179 \\ &= 21.6736 - 19.3179 = 2.3557. \end{aligned}$$

Hence, $x = 226.8$

Hence, the height of the building is 226.8 m approximately.

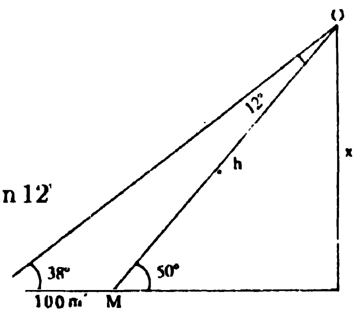


Fig 15.11

Example 15.16

Two ships leave a port at the same time. One goes 24 km per hour in the direction N 38° E and the other travels 32 km per hour in the direction S 52° E. Find the distance between the ships at the end of 3 hours.

Solution

in Fig.15.12,

note that $\angle AOB = 90^\circ$

Hence, $AB = \sqrt{72^2 + 96^2}$ km

$= 24 \times 5$ km $= 120$ km

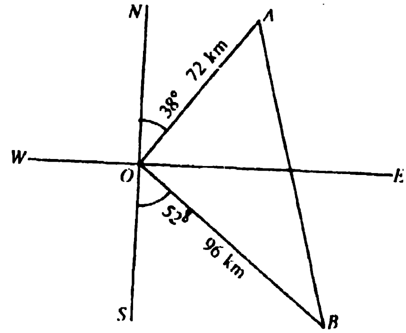


Fig 15.12

Example 15.17

The horizontal distance between two towers is 60 m and the angular depression of the top of the first as seen from the top of the second, which is 150 m high, is 30°. Find the height of the first.

Solution

$BD = 150$m

We have to find h.

Now $$\frac{150-h}{60} = \tan 30^\circ = \frac{1}{\sqrt{3}}$$

or $$h\sqrt{3} = 150\sqrt{3} - 60$$
$$b = 150 - 20\sqrt{3}$$
$$= 115.359$$

Hence, the required height is 115.359 m approximately.

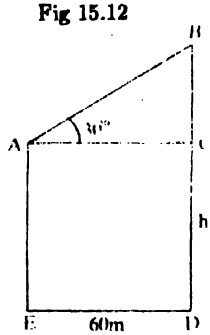


Fig 15.13

EXERCISE 15.9

1. The angle of elevation of a ladder leaning against a house is 58°, and the foot of the ladder is 9.6 m from the house. Find the length of the ladder.
2. A person, standing on the bank of a river, observes that the angle subtended by a tree on the opposite bank is 60°; when he retreats 20 m from the bank, he finds the angle to be 30°. Find the height of the tree and the breadth of the river.
3. From a tower 128 m high, the angle of depression of a car is 26°10′. Find how far the car is from the tower.
4. At a point A, the angle of elevation of a tower is such that its tangent is $\frac{5}{12}$; on walking 240 m nearer the tower the tangent of the angle of elevation is $\frac{3}{4}$. Find the height of the tower.
5. To find the distance between two points A and B on opposite sides of a river, a surveyor runs along a line AC perpendicular to AB. By measurement, he finds $AC = 200$ m and $\angle ACB = 48°40'$. Find the distance AB.
6. Find the height of chimney when it is found that, on walking towards it 50 m in a horizontal line through its base, the angular elevation of its top changes from 30° to 45°

7. A town B is 13 km South and 18 km West of a town A. Find the bearing and distance of B from A.

8. An observer on the top of a cliff 200 m above the sea-level, observes the angles c depressions of two ships on opposite sides of the cliff to be 45° and 30° respectiv .ly. Find the distance between the ships if the line joining them points to the base of :he cliff.

9. The upper part of a tree broken over by the wind makes an angle of 30° with the ground, and the distance from the root to the point where the top of the tree touches the ground is 10 m. What was the height of the tree?

10. From a tower 126 m high, the angles of depression of two rocks which are in a horizontal line through the base of the tower are 16° and 12°20′. Find the distance between the rocks if they are on (a) the same side of the tower; (b) opposite sides of the tower.

11. At the foot of a mountain the elevation of its summit is 45°; after ascending 1000 m towards the mountain up a slope of 30° inclination, the elevation is found to be 60°. Find the height of the mountain.

CHAPTER 16

Inverse Trigonometric Functions

16.1 The Inverse of a Function

Let $f : X \to Y$ be a function. Recall that f is said to be *one-to-one* if $f(x_1) \neq f(x_2)$, whenever $x_1 \neq x_2$. We say that f is *onto* if for each $y \in Y$, there exists an $x \in X$ such that $y = f(x)$. If $f : X \to Y$ is one-to-one and onto then, we can define a unique function $g : Y \to X$ such that $g(y) = x$, where $x \in X$ is such that $y = f(x)$. Thus the domain of g = range of f and range of g = domain of f. The function g is called the inverse of f and is denoted by f^{-1}. Note that the graphs of f and f^{-1} are

$$\{(x, f(x)) | x \in \text{domain} f\} \quad \text{and}$$
$$\{(f(x), x) | x \in \text{domain} f\} \quad \text{respectively.}$$

Example 16.1
Let $f(x) = 2x, 0 \leq x \leq 1$.
Then $f^{-1}(x) = \frac{x}{2}$, $0 \leq x \leq 2$. The student is advised to verify it.

Inverse of Trigonometric Functions

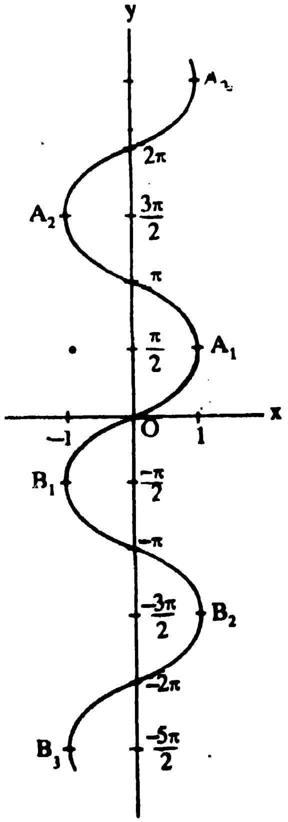
Let us consider the function $f(x) = \sin x$. We know that if $\sin x = \sin\theta$, then

$$x = n\pi + (-1)^n\theta \quad \text{for some integer n.}$$

However, if we restrict the function $\sin x$ to $\frac{-\pi}{2} \leq x \leq \frac{\pi}{2}$, then this function is one-to-one and its image is $-1 \leq y \leq 1$. Actually, $\sin x$ restricted to any of the intervals $\frac{-\pi}{2} \leq x \leq \frac{\pi}{2}$, $\frac{3\pi}{2} \leq x \leq \frac{5\pi}{2}$, $-\frac{5\pi}{2} \leq x \leq -\frac{3\pi}{2}$ etc. is one-to-one and its image is $-1 \leq y \leq 1$. We, therefore, conclude that in each of these intervals we can define the inverse of the sine function. Note that $y = \sin^{-1} x$ means $x = \sin y$. Furthermore, $\sin^{-1} x$ should not be confused with $(\sin x)^{-1}$ which is equal to $\frac{1}{\sin x}$. Also, note that $\sin^{-1} x$ is an angle whose sine is x.

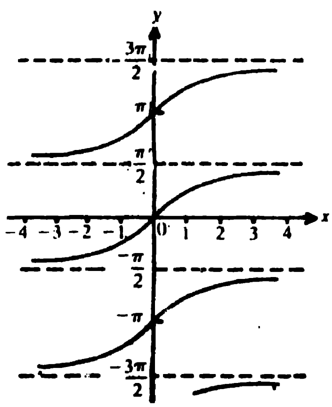
We emphasise that $\sin^{-1} x$ is a function whose domain is $-1 \leq x \leq 1$ and whose range is $\frac{-\pi}{2} \leq y \leq \frac{\pi}{2}$ or $\frac{-5\pi}{2} \leq y \leq \frac{-3\pi}{2}$ or $\frac{3\pi}{2} \leq y \leq \frac{5\pi}{2}$ and so on. Corresponding

to each such interval, we say that we get a branch of the function $\sin^{-1} x$. Thus A_1A_2, A_2A_3 etc. and A_1B_1, B_1B_2 etc. are different branches of $\sin^{-1} x$. Of course these are graphs of different functions.



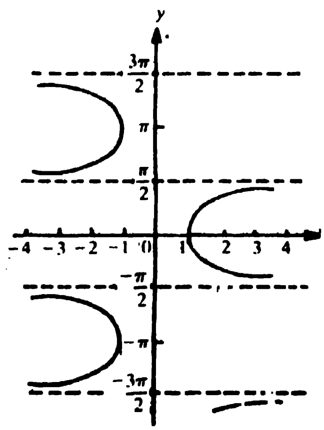
Graph of $y = \sin^{-1} x$

Fig 16.1



Graph of $y = \tan^{-1} x$

Fig 16.2



Graph of $y = \sec^{-1} x$

Fig 16.3

From the graph, we see that for each x, $-1 \leq x \leq 1$, there exists a unique $y \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ such that $\sin y = x$. This y is called the principal value of $\sin^{-1} x$ Notice that $\sin^{-1} : [-1, 1] \rightarrow \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ becomes a function. We also note that the principal value of $\sin^{-1} x$ is numerically least among all values of $\sin^{-1} x$. The considerations similar to the above are valid for other inverse trigonometric functions like $\cos^{-1} x$, $\tan^{-1} x$ and so on.

Remark

The principal value of $\sin^{-1} x$, for $x > 0$, is precisely the length of the arc of unit circle centred at origin which subtends an angle at the centre whose sine is x. For this reason $\sin^{-1} x$ is also denoted by $\arcsin x$. Similarly, $\cos^{-1} x$, $\tan^{-1} x$ etc. arc denoted by $\arccos x$, $\arctan x$ and so on. Following is a table giving the inverse trigonometric functions and their principal value branches:

Functions	*Principal Value Branches*	
$y = \sin^{-1} x$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$,	where $-1 \le x \le 1$
$y = \cos^{-1} x$	$0 \le y \le \pi$,	where $-1 \le x \le 1$
$y = \tan^{-1} x$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$,	where $-\infty < x < \infty$
$y = \sec^{-1} x$	$0 \le y < \frac{\pi}{2}$,	where $1 \le x < \infty$
	and	
	$\frac{\pi}{2} < y \le \pi$,	where $-\infty < x \le -1$
$y = \operatorname{cosec}^{-1} x$	$-\frac{\pi}{2} \le y < 0$,	where $-\infty < x \le -1$
	and	
	$0 < y \le \frac{\pi}{2}$,	where $1 \le x < \infty$
$y = \cot^{-1} x$	$0 < y < \pi$,	where $-\infty < x < \infty$

Note

1. When y is positive $(0 < y \le 1)$, there are two angles, one between 0 and $\frac{\pi}{2}$ and the other between $\frac{-\pi}{2}$ and 0 having their cosines equal to y (Recall $\cos x$ is an even function). In this case, we take the angle lying between 0 and $\frac{\pi}{2}$ as it lies in the principal value branch of $\cos^{-1} x$.

   For example, $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)=\frac{\pi}{6}$ and not $-\frac{\pi}{6}$ even though $\cos\left(-\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2}$
2. Whenever no branch of an inverse trigonometric function is specifically mentioned, we mean the principal branch of the function.
3. We have given the graphs of $\sin^{-1} x$. The graphs of other inverse trigonometric functions can be drawn from the knowledge of the graphs of corresponding trigonometric functions. The graphs of $\tan^{-1} x$ and $\sec^{-1} x$ are given in Figs. 16.2 and 16.3, respectively.

16.2 Properties of Inverse Trigonometric Functions

1. $\theta = \sin^{-1}(\sin\theta)$ for $\frac{-\pi}{2} \le \theta \le \frac{\pi}{2}$. Let $\sin\theta = x$ and $\frac{-\pi}{2} \le \theta \le \frac{\pi}{2}$. Then $\theta = \sin^{-1} x = \sin^{-1}(\sin\theta)$.

   Similarly, $\theta = \cos^{-1}(\cos\theta),\ 0 \le \theta \le \pi$

$$= \tan^{-1}(\tan\theta),\ -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$
$$= \sec^{-1}(\sec\theta),\ 0 \le \theta < \frac{\pi}{2},\ \frac{\pi}{2} < \theta \le \pi$$
$$= \operatorname{cosec}^{-1}(\operatorname{cosec}\theta),\ 0 < \theta < \frac{\pi}{2},\ -\frac{\pi}{2} \le \theta < 0$$
$$= \cot^{-1}(\cot\theta),\ 0 < \theta < \pi$$

Note that in the above formulas and the others to follow, assume that we are dealing with the principal branches of the functions. In no case, we should consider two different branches simultaneously while working with these formulas.

For example, $\sin\left(\frac{3\pi}{4}\right)=\frac{1}{\sqrt{2}}$

Also, $\sin\left(\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}$

But $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$ is $\frac{\pi}{4}$ and *not* $\frac{3\pi}{4}$, since $\frac{3\pi}{4}$ does not lie within $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$, the principal value branch of $\sin^{-1} x$.

$$\text{Hence, } \sin^{-1}\left(\sin\frac{3\pi}{4}\right)=\sin^{-1}\left[\sin\left(\pi-\frac{\pi}{4}\right)\right] = \sin^{-1}\left[\sin\frac{\pi}{4}\right]=\frac{\pi}{4}$$

2. $\operatorname{cosec}^{-1} x = \sin^{-1}\frac{1}{x}$, if $x > 0$, $\cot^{-1} x = \tan^{-1}\frac{1}{x}$ if $x > 0$, $\cot^{-1} x = \pi + \tan^{-1}\frac{1}{x}$ if $x < 0$, and $\sec^{-1} x = \cos^{-1}\frac{1}{x}$ if $x > 0$. Let us prove, for example, $\operatorname{cosec}^{-1} x = \sin^{-1}\frac{1}{x}$ if $x > 0$. Let $\operatorname{cosec}^{-1} x = y$. Then $x = \operatorname{cosec} y$, $\frac{-\pi}{2} \le y \le \frac{\pi}{2}$ and $y \ne 0$. So $\frac{1}{x} = \sin y$ or $y = \sin^{-1}(\frac{1}{x})$. Other formulas can be similarly proved.

3. Since $\sin^{-1} x$, $\tan^{-1} x$ and $\operatorname{cosec}^{-1} x$ may assume positive or negative values, we determine the value of $\sin^{-1}(-x)$. Let $\sin^{-1}(-x) = y$. Then $\frac{-\pi}{2} \le y \le \frac{\pi}{2}$ and $-x = \sin y$ or $x = -\sin y = \sin(-y)$.
Hence, $-y = \sin^{-1} x$ or $y = -\sin^{-1} x$.

   Hence, $\sin^{-1}(-x) = -\sin^{-1} x$
   We can similarly prove that $\cos^{-1}(-x) = \pi - \cos^{-1} x$, $\tan^{-1}(-x) = -\tan^{-1} x$ and $\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1} x$ and $\cot^{-1}(-x) = \pi - \cot^{-1} x$.

4. We now prove the following important relations between the inverse trigonometric functions:

   (i) $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$

   (ii) $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$

   (iii) $\operatorname{cosec}^{-1} x + \sec^{-1} x = \frac{\pi}{2}$

   (iv) $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\frac{x+y}{1-xy}$, if $xy < 1$

   (v) $\tan^{-1} x - \tan^{-1} y = \tan^{-1}\frac{x-y}{1+xy}$, if $xy > -1$

   (vi) $2\tan^{-1} x = \sin^{-1}\frac{2x}{1+x^2} = \tan^{-1}\frac{2x}{1-x^2}$, if $|x| < 1$.

Proof

(i) Let $\sin^{-1} x = \theta$. Then $\frac{-\pi}{2} \leq \theta \leq \frac{\pi}{2}$ and $x = \sin\theta = \cos(\frac{\pi}{2} - \theta)$, $0 \leq \frac{\pi}{2} - \theta \leq \pi$, $\frac{\pi}{2} - \theta = \cos^{-1} x$.

Hence, $\sin^{-1} x + \cos^{-1} x = \theta + \frac{\pi}{2} - \theta = \frac{\pi}{2}$

(ii) and (iii) are similarly proved.

(iv) $\tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x+y}{1-xy}$.

Let $\tan^{-1} x = \theta_1$, $\tan^{-1} y = \theta_2$. Then $x = \tan\theta_1$, $y = \tan\theta_2$, $\frac{-\pi}{2} < \theta_1, \theta_2 < \frac{\pi}{2}$.

Now, $\tan(\theta_1 + \theta_2) = \frac{\tan\theta_1 + \tan\theta_2}{1 - \tan\theta_1 . \tan\theta_2}$

$$= \frac{x+y}{1-xy}.$$

Hence, $\theta_1 + \theta_2 = \tan^{-1} \frac{x+y}{1-xy}$ if $\frac{-\pi}{2} < \theta_1 + \theta_2 < \frac{\pi}{2}$.

Note that if $xy > 1$, then (iv) cannot hold. If $xy = 1$ then R.H.S of (iv) is not defined. Suppose $xy > 1$. If both $x, y > 0$, then $\frac{x+y}{1-xy} < 0$ and R.H.S of (iv) is negative, while L.H.S. is positive. If both $x, y < 0$, then $\frac{x+y}{1-xy} > 0$ and R.H.S of (iv) is positive while L.H.S is negative. Therefore, (iv) cannot hold if $xy \geq 1$. However, if $xy < 1$, then we can show that $\theta_1 + \theta_2$ lies between $\frac{-\pi}{2}$ and $\frac{\pi}{2}$ (Problem 14, Exercise 16.1).

Hence, in this case (iv) holds.

(v) We prove (v) in the same way as (iv). We need only to change y to $-y$.

(vi) Let $\tan^{-1} x = \theta$, then $x = \tan\theta$.

Now, $\frac{2x}{1+x^2} = \frac{2\tan\theta}{1+\tan^2\theta}$

$$= \frac{2\sin\theta\cos\theta}{\sin^2\theta + \cos^2\theta} = \sin 2\theta$$

Hence, $2\theta = \sin^{-1} \frac{2x}{1+x^2}$ if $\frac{-\pi}{2} \leq 2\theta \leq \frac{\pi}{2}$

Similarly, we can prove the other relations. We must remark that these results hold only for restricted values of x. Thus $\sin^{-1} \frac{2x}{1+x^2} = 2\tan^{-1} x$ if $-1 \leq x \leq 1$. Notice that if $x > 1$ or $x < -1$, then $2\tan^{-1} x$ is either $> \frac{\pi}{2}$ or $< \frac{-\pi}{2}$. Similarly,

$$\cos^{-1} \frac{1-x^2}{1+x^2} = 2\tan^{-1} x \text{ if } 0 \leq x < \infty \text{ and } 2\tan^{-1} x = \tan^{-1} \frac{2x}{1-x^2} \text{ if } -1 < x < 1.$$

Example 16.2
Find the principal values of

(i) $\text{cosec}^{-1}(-1)$

(ii) $\cot^{-1}(\frac{-1}{\sqrt{3}})$.

Solution

(i) Let $\text{cosec}^{-1}(-1) = y$. then y must satisfy $\frac{-\pi}{2} \leq y < 0$ and $\text{cosec}\, y = -1$. This is true only for $y = \frac{-\pi}{2}$, which is, therefore, the principal value of $\text{cosec}^{-1}(-1)$.

(ii) Let $\cot^{-1}(\frac{-1}{\sqrt{3}}) = y$. Then $\cot y = \frac{-1}{\sqrt{3}}$ or $\tan y = -\sqrt{3}$. Now, $\tan\theta = -\tan(\pi - \theta)$. Therefore, since $\tan\frac{\pi}{3} = \sqrt{3}$, we must have principal value of $\cot^{-1}(\frac{-1}{\sqrt{3}})$ as $\pi - \frac{\pi}{3} = \frac{2\pi}{3}$.

Example 16.3
Find two branches, other than the principal value branch of $\tan^{-1} x$.

Solution
By graph of $\tan x$, we can see that the inverse function of $\tan x$ exists in each of the intervals $\frac{-3\pi}{2} < y < \frac{-\pi}{2}$, $\frac{\pi}{2} < y < \frac{3\pi}{2}$, $\frac{3\pi}{2} < y < \frac{5\pi}{2}$ and so on. Hence, the two branches of $\tan^{-1} x$ can be taken to be

$$\frac{-3\pi}{2} < \tan^{-1} x < \frac{-\pi}{2} \text{ and } \frac{\pi}{2} < y < \frac{3\pi}{2}.$$

Example 16.4
Prove that

$$2\tan^{-1}(\frac{1}{x}) = \sin^{-1}\frac{2x}{x^2+1} \text{if} \, |x| \geq 1$$

Solution
Let $\tan^{-1}\frac{1}{x} = \theta$.
Then $\frac{1}{x} = \tan\theta, |\theta| \leq \frac{\pi}{4}$ since $\left|\frac{1}{x}\right| \leq 1$.
Hence, L.H.S. $= 2\theta$

$$\text{R.H.S.} = \sin^{-1}\frac{2x}{x^2+1} = \sin^{-1}\frac{\frac{2}{x}}{1+\frac{1}{x^2}}$$

$$\begin{aligned} &= \sin^{-1}\frac{2\tan\theta}{1+\tan^2\theta} = \sin^{-1}(\sin 2\theta) \\ &= 2\theta = \text{L.H.S.} \end{aligned}$$

Example 16.5
Prove that

$$\tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{13} = \tan^{-1}\frac{2}{9}.$$

Solution
Let $\tan^{-1}\frac{1}{7} = \theta_1$ and $\tan^{-1}\frac{1}{13} = \theta_2$. Then

$$\begin{aligned} \frac{1}{7} + \frac{1}{13} &= \tan\theta_1 + \tan\theta_2. \\ &= \tan(\theta_1 + \theta_2)(1 - \tan\theta_1 . \tan\theta_2) \\ &= \tan(\theta_1 + \theta_2)\left(1 - \frac{1}{7}.\frac{1}{13}\right) \end{aligned}$$

or $$\frac{20}{91} = \frac{90}{91}\tan(\theta_1 + \theta_2)$$

or $$\tan(\theta_1 + \theta_2) = \frac{2}{9}$$

or $$\theta_1 + \theta_2 = \tan^{-1}\frac{2}{9}$$

Alternately,

$$\begin{aligned} &\tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{13} \\ &= \tan^{-1}\frac{\frac{1}{7} + \frac{1}{13}}{1 - \frac{1}{7}\cdot\frac{1}{13}} \\ &= \tan^{-1}\frac{2}{9} \end{aligned}$$

Example 16.6
Solve $\tan^{-1}2x + \tan^{-1}3x = \frac{\pi}{4}$.

Solution

$$\begin{aligned} &\tan^{-1}2x + \tan^{-1}3x \\ &= \tan^{-1}\frac{2x + 3x}{1 - 2x.3x} \\ &= \tan^{-1}\frac{5x}{1 - 6x^2}, \text{ provided } \quad 6x^2 < 1. \end{aligned}$$

Hence $\tan^{-1}\dfrac{5x}{1-6x^2} = \dfrac{\pi}{4}.$

or $\dfrac{5x}{1-6x^2} = 1$

or $6x^2+5x-1 = 0$ or $(6x-1)(x+1)=0$

Hence, the required solution is $x = \frac{1}{6}$

Note that $x=-1$ does not satisfy the solution as $\tan^{-1}(-2)+\tan^{-1}(-3)<0$.

Example 16.7

Write $\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)$ in the simplest form.

Solution

Let $\tan^{-1}\frac{\cos x}{1+\sin x} = \theta.$

Then $\frac{\cos x}{1+\sin x} = \tan\theta$

Now,

$$\frac{\cos x}{1+\sin x} = \frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2} = \frac{\cos\frac{x}{2}-\sin\frac{x}{2}}{\cos\frac{x}{2}+\sin\frac{x}{2}}$$

$$= \frac{\sin\left(\frac{\pi}{4}-\frac{x}{2}\right)}{\cos\left(\frac{\pi}{4}-\frac{x}{2}\right)} = \tan\left(\frac{\pi}{4}-\frac{x}{2}\right) = \tan\theta.$$

Hence $\theta = \frac{\pi}{4}-\frac{x}{2}$

or $\tan^{-1}(\frac{\cos x}{1+\sin x}) = \frac{\pi}{4}-\frac{x}{2}$ if $\frac{-\pi}{2} < \frac{\pi}{4}-\frac{x}{2} < \frac{\pi}{2}$, i.e. if $\frac{-\pi}{2} < x < \frac{3\pi}{2}$

Example 16.8

Show that

$$\tan^{-1}\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}} = \frac{\pi}{4}+\frac{1}{2}\cos^{-1}x^2.$$

Solution

Let $\theta = \frac{1}{2}\cos^{-1}x^2, x\neq 0$. Then $x^2=\cos 2\theta, 0\leq 2\theta<\frac{\pi}{2}$. Hence

$$\begin{aligned}\text{L.H.S.} &= \tan^{-1}\left(\frac{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}\right)\\ &= \tan^{-1}\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)\\ &= \tan^{-1}\left(\frac{1+\tan\theta}{1-\tan\theta}\right)\end{aligned}$$

$$= \tan^{-1}\left(\tan(\frac{\pi}{4}+\theta)\right) = \frac{\pi}{4}+\theta \left(\text{since} \quad \frac{\pi}{4} \le \frac{\pi}{4}+\theta < \frac{\pi}{4}+\frac{\pi}{4}\right)$$
$$= \frac{\pi}{4}+\frac{1}{2}\cos^{-1} x^2$$
$$= \text{R.H.S.}$$

EXERCISE 16.1

1. Find the principal values of

   (i) $\sin^{-1}(-1)$

   (ii) $\tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)$

   (iii) $\cos^{-1}\left(\frac{-1}{2}\right)$

   (iv) $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$

   (v) $\cot^{-1}(\sqrt{3})$

   (vi) $\text{cosec}^{-1}(-2)$

2. Find any three branches, other than the principal value branch, of each of the following:

   (i) $\sec^{-1} x$

   (ii) $\text{cosec}^{-1} x$

   (iii) $\cot^{-1} x$

Prove the following:

3. $\cos^{-1} x = 2\sin^{-1}\sqrt{\frac{1-x}{2}} = 2\cos^{-1}\sqrt{\frac{1+x}{2}}$
4. $3\cos^{-1} x = \cos^{-1}(4x^3 - 3x)$
5. $\sin(2\sin^{-1} x) = 2x\sqrt{1-x^2}$
6. $\tan^{-1}\sqrt{x} = \frac{1}{2}\cos^{-1}\frac{1-x}{1+x}$
7. $\tan^{-1} t + \tan^{-1}\frac{2t}{1-t^2} = \tan^{-1}\frac{3t-t^3}{1-3t^2}, t^2 < \frac{1}{3}$
8. $\tan^{-1}\frac{1}{4} + \tan^{-1}\frac{2}{9} = \frac{1}{2}\cos^{-1}\frac{3}{5}$
9. $\cot^{-1}\frac{ab+1}{a-b} + \cot^{-1}\frac{bc+1}{b-c} + \cot^{-1}\frac{ca+1}{c-a} = 2\pi$ if $a < b < c$

10. If $\cos^{-1}\frac{x}{a} + \cos^{-1}\frac{y}{b} = \alpha$, prove that

$$\frac{x^2}{a^2} - \frac{2xy}{ab}\cos\alpha + \frac{y^2}{b^2} = \sin^2\alpha$$

11. Find the value of

$$\tan\left(\frac{1}{2}\sin^{-1}\frac{2x}{1+x^2} + \frac{1}{2}\cos^{-1}\frac{1-y^2}{1+y^2}\right)$$

12. Solve the following equations:

(i) $\tan^{-1}\frac{x-1}{x-2} + \tan^{-1}\frac{x+1}{x+2} = \frac{\pi}{4}$

(ii) $2\tan^{-1}(\cos x) = \tan^{-1}(2\operatorname{cosec} x)$.

13. Write the following in the simplest form:

(i) $\tan^{-1}\left(\frac{x}{\sqrt{a^2-x^2}}\right)$

(ii) $\tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right)$

(iii) $\tan^{-1}\left(\frac{\cos x - \sin x}{\cos x + \sin x}\right)$

[illegible]an$^{-1}\left(\frac{3a^2x - x^3}{a(a^2-3x^2)}\right)$

[illegible] $= \theta_1$ and $\tan^{-1} y = \theta_2$. Show that $\frac{-\pi}{2} < \theta_1 + \theta_2 < \frac{\pi}{2}$ provided $xy < 1$.
[Hint: If θ [illegible] 0 and $\theta_2 < 0$ or $\theta_1 < 0$ and $\theta_2 > 0$, then $\frac{-\pi}{2} < \theta_1 + \theta_2 < \frac{\pi}{2}$, since [illegible]$\frac{}{2} < \theta_1 < \frac{\pi}{2}$ and $\frac{-\pi}{2} < \theta_2 < \frac{\pi}{2}$.
Suppose $\theta_1 > 0, \theta_2 > 0$. Then $xy < 1 \Rightarrow x < \frac{1}{y}$ or $\tan\theta_1 < \frac{1}{\tan\theta_2}$.
Hence, $\tan\theta_1 < \cot\theta_2 = \tan\left(\frac{\pi}{2} - \theta_2\right)$

Since, $\tan\theta$ is increasing in the interval $\left(0, \frac{\pi}{2}\right)$, we get $\theta_1 < \frac{\pi}{2} - \theta_2$ or $\theta_1 + \theta_2 < \frac{\pi}{2}$.
Since $\theta_1 + \theta_2 > 0$, we have $\frac{-\pi}{2} < \theta_1 + \theta_2 < \frac{\pi}{2}$. If $\theta_1 < 0$ and $\theta_2 < 0$, then $\theta_1 + \theta_2 < 0$.
Also $xy < 1 \Rightarrow \tan\theta_1 \tan\theta_2 < 1$
or $(-\tan\theta_1)(-\tan\theta_2) < 1$.
Hence, $\tan(-\theta_1)\tan(-\theta_2) < 1$. but $-\theta_1 > 0$ and $-\theta_2 > 0$.

Hence from above, we get $(-\theta_1) + (-\theta_2) < \frac{\pi}{2}$ or $\theta_1 + \theta_2 > \frac{-\pi}{2}$. This implies the result.]

MISCELLANEOUS EXERCISES

1. If $x+y+z=xyz$, prove that
$$\frac{2x}{1-x^2}+\frac{2y}{1-y^2}+\frac{2z}{1-z^2}=\frac{8xyz}{(1-x^2)(1-y^2)(1-z^2)}.$$

2. If $\frac{\sin\alpha}{a}=\frac{\cos\alpha}{b}$, prove that $a\sin 2\alpha+b\cos 2\alpha=b$.

3. If $\tan(\alpha+\theta)=n\tan(\alpha-\theta)$, show that $(n+1)\sin 2\theta=(n-1)\sin 2\alpha$.

4. If $\sin\alpha+\sin\beta=a$, and $\cos\alpha+\cos\beta=b$, show that

   (i) $\sin(\alpha+\beta)=\dfrac{2ab}{a^2+b^2}$

   (ii) $\cos(\alpha+\beta)=\dfrac{b^2-a^2}{b^2+a^2}$.

5. Solve the equation: $\cos\theta+\cos 2\theta+\cos 3\theta=0$

6. If α, β are two different values of θ lying between 0 and 2π which satisfy the equation $6\cos\theta+8\sin\theta=9$, find the value of $\sin(\alpha+\beta)$.

7. If $\cos^{-1}x+\cos^{-1}y+\cos^{-1}z=\pi$, show that $x^2+y^2+z^2+2xyz=1$

8. Solve: $\sin^{-1}x+\sin^{-1}2x=\frac{\pi}{3}$

9. In any triangle prove that $(a+b+c)(\tan\frac{A}{2}+\tan\frac{B}{2})=2c\cot\frac{C}{2}$.

10. Prove that $\dfrac{(a+b+c)^2}{a^2+b^2+c^2}=\dfrac{\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}}{\cot A+\cot B+\cot C}$.

11. In a ΔABC, if $c=60°$, then prove that $\frac{1}{a+c}+\frac{1}{b+c}=\frac{3}{a+b+c}$.

12. Prove that, in a ΔABC, $\Delta=4Rr\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$.

13. Prove that $\Delta=2R^2\sin A\sin B\sin C$.

14. Prove that, in any ΔABC, $\frac{1}{bc}+\frac{1}{ca}+\frac{1}{ab}=\frac{1}{2Rr}$.

15. Over a tower AB of height h metres there is a flag staff BC. AB and BC are making equal angles at a point distant d metres from the foot A of the tower. Show that the height of the flag staff is
$h\left[\dfrac{d^2+h^2}{d^2-h^2}\right]$ metres.

16. A tower subtends an angle α at a point A on the same level as the foot of the tower B is a point vertically above A and AB is h metres. The angle of depression of the foot of the tower measured from B is β. Show that height of the tower is $h\tan\alpha\cot\beta$.

17. A person observes the angle of elevation of the peak of a hill from a station to be α. He walks C metres along a slope inclined at the angle β and finds the angle of elevation of the peak of the hill to be γ. Show that the height of the peak above the ground is $c \sin \alpha(\gamma - \beta)/\sin(\gamma - \alpha)$.

18. Show that the lines $2x^2+6xy+y^2 = 0$ are equally inclined to the lines $4x^2+18xy+y^2 = 0$.

19. If one of the straight lines given by the equation $ax^2 + 2hxy + by^2 = 0$ coincide with one of those given by $a'x^2 + 2h'xy + b'y^2 = 0$, and the other lines represented by them be perpendicular, prove that

$$\frac{ha'b'}{b'-a'} = \frac{h'ab}{b-a} = \frac{1}{2}\sqrt{-aa'bb'}$$

20. Prove that

$$\frac{2\cos 2^n\theta + 1}{2\cos\theta + 1} = (2\cos\theta - 1)(2\cos 2\theta - 1)(2\cos 2^2\theta - 1)..(2\cos 2^{n-1}\theta - 1)$$

21. Prove that

$$\sin\frac{\pi}{5}\sin\frac{2\pi}{5}\sin\frac{3\pi}{5}\sin\frac{4\pi}{5} = \frac{5}{16}$$

22. Solve: $\sin x + \cos x = \sqrt{2}\cos A$.

CHAPTER 17

Frequency Tables

17.1 Raw Data

Statistics deals with information given in numerical terms, for example, the amount of land owned by different peasants in a village, tehsil or district, the monthly wage earned by workers in a factory, the number of male and female voters in a constituency, the number of people of different religions in a town, etc.

Such information is collected through *censuses* and *surveys*. The Government of India, for example, conducts an All India Census every ten years (the last was in 1991) to record the actual number of persons alive at a given time, along with their age, sex, occupation, housing condition, etc. The *National Sample Survey* (or NSS) is a programme of periodically obtaining information, in which the Government is interested, to help it in framing its policies. For example, the NSS may be asked to collect information about the extent of unemployment, the requirement for housing, etc. The Ministry of Labour conducts *family budget surveys* to find out how much an average worker's family has to spend to obtain the minimum requirements of food, clothing, housing, education, medical aid, etc. This information is then used to calculate the *cost of living index* which measures the effect of changing prices on the standard of living.

The terms *census* and *survey* are used to distinguish between two different techniques of collecting information. The All India Census information about age, sex, education, etc., is obtained about *every* person alive at that time. It is another matter that we may not succeed in covering the entire population, but every effort is made to do so. The Ministry of Labour does not attempt to contact *every* working class family to get the required information for calculating the cost of living index. Instead, it *selects* a group of workers to represent all the workers and then collects the information on family expenses from the workers in the selected group only.

The term *census* is used to indicate that information is collected from *all* the members of the group in which we are interested. The term survey (or *sample survey*) indicates that the information is collected only from some selected members of the group and not from all of them.

In addition to these two methods, we also come across regular collection and recording

of information in a routine manner. The Meteorological Department maintains a daily record of maximum and minimum temperatures, rainfall, relative humidity, atmospheric pressure, etc. This information is used for weather forecasting. The Indian Railways keep a daily record of movement of passengers and goods, the income earned through fares and freights, etc. The Ministry of Commerce keeps a record of the quantity and value of goods imported and exported every month.

The information collected through censuses and surveys, or in a routine manner, is called *raw data*. The word *data* means information (its exact dictionary meaning is : given facts.) The adjective raw attached to it indicates that the information thus collected and recorded cannot be put to any use immediately and directly but has to be *processed*, that is, converted to a more suitable form, before it begins to make sense to be utilised gainfully. Just as raw rice has to be cooked before it can be eaten and digested, the raw data too have to be converted into another form before any use can be made of it.

Take the All India Census, for example. The census enumerators go to every household and collect information about all the members. If we are interested in knowing how old the different persons alive at the time of the census were, our raw data coming from census records is simply a string of numbers: 11, 3, 42, 37, ... Each number stands for the age of a person, there being one number for each person alive at the time of the census. As per the 1971 census, the population of India exceeds 600 million and so our raw data is a string of more than 600 million numbers. In a sense this raw data has *all* the information about the ages of different persons on a given date. In another sense, it provides no information at all, because when we look at it we are simply bewildered by the very size of the data which has been collected. The Registrar General's office a part of the census organisation — processes this raw data and presents it in form of different tables which make it easy to understand the information provided by the raw data. One such table is given below.

TABLE 17.1

Table C-II: Age and Marital Status (Uttar Pradesh)

Age-group	*Total Rural Urban*	*Persons*	*Male*	*Female*
All Ages	T	88,341,144*	47,016,421	41,324,723
	R	75,952,548	40,214,012	35,738,536
	U	12,388,596	6,802,409	5,586,187
0-9	T	26,105,403	13,724,165	12,381,238
	R	22,626,315	11,914,955	10,711,360
	U	3,479,088	1,809,210	1,669,878
10-14	T	10,859,933	6,061,626	4,798,307
	R	9,243,379	5,195,347	4,048,032
	U	1,616,554	866,279	750,275

Age-group	*Total Rural Urban*	*Persons*	*Male*	*Female*
15-19	T	7,184,548	3,978,135	3,206,413
	R	5,960,357	3,293,197	2,667,160
	U	1,224,191	684,938	539,253
20-24	T	6,531,468	3,246,939	3,284,529
	R	5,432,600	2,633,426	2,799,174
	U	1,098,868	613,513	485,355
25-29	T	6,474,870	3,275,180	3,199,690
	R	5,535,799	2,759,930	2,775,869
	U	939,071	515,250	423,821
30-34	T	5,910,572	3,013,133	2,897,439
	R	5,106,844	2,563,939	2,542,905
	U	803,728	449,194	354,534
35-39	T	5,174,221	2,719,620	2,454,601
	R	4,463,129	2,322,440	2,140,689
	U	711,092	397,180	313,912
40-44	T	4,737,880	2,550,573	2,187,307
	R	4,094,263	2,177,784	1,916,479
	U	643,617	372,789	270,828**
45-49	T	3,676,085	1,984,626	1,691,459
	R	3,186,065	1,693,346	1,492,719
	U	490,020	291,280	198,740
50-54	T	3,598,058	2,046,915	1,551,143
	R	3,141,675	1,772,218	1,369,457
	U	456,383	274,697	181,686
55-59	T	2,103,947	1,125,033	978,914
	R	1,853,821	978,649	875,172
	U	250,126	146,384	103,742
60-64	T	2,636,013	1,459,816	1,176,197
	R	2,336,025***	1,289,280	1,046,745
	U	299,988	170,536	129,452
65-69	T	1,241,876	681,421	560,455
	R	1,103,755	602,462	501,293
	U	138,121	78,959	59,162

Age-group	Total Rural Urban	Persons	Male	Female
70+	T	2,099,009	1,145,516	953,493**
	R	1,861,574	1,013,535	848,039
	U	237,435	131,981	105,454
Age not stated	T	7,261	3,723	3,538
	R	6,947	3,504	3,443
	U	314	219	95

Source: Table C-II, p.48, Census of India, 1971 (Uttar Pradesh), series 21, part II-C(ii)
*printed as 88,319,144
**printed as 270,823
***printed as 2,136,025
**** printed as 953,423

Table 17.1 gives the total population of Uttar Pradesh as 88,341,144 of which 47,016,421 (53.2%) are males and 41,324,723 (46.8%) are females. Nearly 86% (75,952,548/88,341,144) of the population, lives in rural areas and only 14% are in urban areas.

The number of persons falling in different age-groups, that is, with age between 0 to 9 years, between 10 years and 14 years, and so on, are shown separately. At the end of the table we have the number of persons whose ages were not properly recorded at the time of the census. The total of such persons is 7,261, that is merely 0.008% of the whole population which shows the care and pains taken to record the information at the time of the census.

Compared to the raw data on ages, this table provides a very clear picture. We see, after making some calculations, that 57,156,222 persons (64.7%) are below 30 years in age and 11,678,903 (13.2%) are 50 years or more in age. We can also obtain from the table the percentage of persons belonging to the different age-groups, that is the *age-distribution* of the population, the *sex-ratio* (ratio of female to males) in the different age-groups, and so on. For example, out of all the males $\frac{13,724,165}{47,016,421} = 29\%$ are in the age-group 0–9, 13% in the age-group 10–14, and so on; the ratio of females to males in the age-group 30–34 is $\frac{2,542,905}{2,563,939} = 99\%$ in rural areas, and 79% in urban areas, and so on.

The table as published by the census authorities had some printing errors. We have corrected the errors and given the corrected values in the table. The original values which were in error are given at the foot of the table. Can you see how these errors were discovered and corrected.

As another example, take a look at the table below which gives the death rates for males and females, separately and considered together, in different age-groups.

These death rates are obtained by taking the ratio of the number of persons in a given age-group dying during a year to the total number of persons of that group alive at the beginning of the year. (The actual procedure is slightly different but the essential idea is the same). The ratio is not expressed as a percentage (i.e. per 100 of the population

above) but per 1000 persons alive. Thus the figure 44.2 at the top of column (2) states that out of 1000 males in the age-group 1-4 years alive at the beginning of the year, 44.2 had died by the time the year ended. Most demographic ratios (death rates, birth rates, etc.) are presented in this manner in terms of 1000 units in the denominator and not as percentages. The death rate 44.2 would, when expressed as a percentage, become 4.42.

TABLE 17.2

Age-sex-Specific Death Rates in Rural India (1980)

(Bihar and West Bengal are excluded) (per'000 population)

Age-Groups	*Males*	*Females*	*Both*
(1)	*(2)*	*(3)*	*(4)*
1- 4	44.2	48.1	46.1
5- 9	3.6	4.5	4.0
10-14	1.8	1.8	1.8
15-19	2.1	3.3	2.7
20-24	2.5	4.1	3.3
25-29	2.3	4.6	3.4
30-34	3.5	3.9	3.7
35-39	5.0	5.0	5.0
40-44	7.3	5.8	5.6(*)
45-49	9.7	7.7	8.8
50-54	15.1	10.9	13.1
55-59	22.0	17.3	19.8
60-64	35.9	28.4	32.2
65-69	62.2	43.2	52.6
70+	100.6	87.8	94.0
All ages	13.5	13.6	13.5

Source: Health Statistics of India, 1980 (taken from Table 8.14, *Rural Development Statistics*, NIRD Hyderabad, 1985, pp.309-10)

Columns (2) and (3) give the death rates for males and females respectively, and column (4) gives the death rates for males and females taken together. Incidentally, there is some error in the values given in the row marked with an asterisk(*). Can you see why the values as given cannot be correct?

The raw data for this table would have consisted of a record of all persons alive at the beginning of the year with their sex and age, and a record of all deaths during the year with the sex and age at death. The raw data presented in the form of the table brings out certain interesting features which could not have been noticed by examining the raw data alone.

When we look at the male and female death rates for the whole population (in the row corresponding to 'all ages') we find that the two are not very different. If we had only this information we could have concluded that the chances of dying during the year

are the same for males and females. But when we look at the death rates for different age-groups (i.e., the *age-specific death rates*) we find that the reality is quite different. We find that the death rate for women is more than that of men in age-groups 1-4 and 5-9, and become equal for age-group 10-14. The death rate for women is again higher in age groups 15-19, 20-24, 25-29, 30-34 and then is lower than that for men in age-groups of 40 and above. Can you think of any reasons why it is so?

17.2 Variables of Observation

If we look at the Census example we find that for each person we have recorded the age and sex, and whether the person belongs to a rural or urban area. We say that we have taken observations on three *variables*: age, sex and place of residence. The term variable thus stands for what is being observed. It is called a variable because the results obtained after observing it are different for different persons. Thus one person may have age 11, another 34, and so on. We say that 11,34, ... are the *values* of the age-variable for these persons. Similarly, the values of the sex variable are 'male' and 'female'. Now, the age variable can take any one of the values 1,2,3 ..., though we may not be able to *observe* all of them, for, there may be no person with age 3 in the population at a given time. Similarly, if there are only males in a group which is being observed, only the value 'male' of the sex-variable will appear in our records. But when we think of the sex-variable as such we know that the value 'female' can also occur. In view of this we make a distinction between the *observed values* and the *possible values* of a variable.

A variable is completely described by its descriptive name and the description of all the values it can possibly take. Usually only the name of the variable is given, its possible values being clear from the context.

In the census example we had another variable called 'place of residence' and its possible values were taken as 'rural' and 'urban'. The values of this variable could also have been recorded as the name of a state, district, city, village, etc. Thus two variables may have the same name but the sets of possible values may be different for them. We regard two variables as different if their sets of possible values are different, even if they have the same name.

17.3 Qualitative and Quantitative Variables

In the census example, the values of the age-variable were numbers, those of the sex-variable were not numbers but the names 'male' and 'female' describing a certain type or quality. Similarly, the variable called 'place of residence' also did not have numbers as its possible values, but the names 'rural' and 'urban'.

Variables of observation with numbers as possible values are called *quantitative variables:* those with names of things, places, attributes, etc., as possible values are called *qualitative variables*. A word of caution is necessary here. The possible values of the sex-variable could have been *recorded* as 1 and 2 (1 standing for 'male', 2 for 'female'). The recorded values of this variable would then *appear* as numbers but that

does not make the variable a quantitative variable. What is required for a variable to be called a quantitative variable is not the mere fact that its values are *recorded* as numbers, but that they are *really* numbers on which arithmetic operations can be meaningfully carried out. For example, it makes sense to talk of the sum, product and difference of two ages: but to talk of the sum of 'male' and 'female' values makes no sense even if they have been recorded as 1 and 2.

Age, height, area of a plot of land, weight of a basket of fruit, income of a worker, price of wheat, etc., are examples of quantitative variables. Examples of qualitative variables are sex, religion, caste, colour of hair, etc.

17.4 Units of Observation

We have explained above that a variable can take different values all of which may or may not be actually observed in a given situation. Each *recorded* or *observed* value of a variable is associated with a person, place, object, etc. Thus each recorded value of the age-variable refers to a particular person, each recorded value of land area refers to a particular plot of land, etc. Hence, in each case we have to clarify what each recorded value of a variable refers to or is associated with. The term *unit of observation* will be used to describe what the values of a variable are attached to. If we have the record of the results of a particular examination, the variable of observation could be the marks obtained (a quantitative variable) and the corresponding units of observation would be the students who had appeared in that examination. Each recorded value of the variable would then refer to a particular student.

In the census example, the units of observation are the persons alive at the time of the census and to each unit of observation we associate the value of three variables: age, sex, and place of residence. Thus different variables of observation may be associated with the same unit of observation.

17.5 Frequency Tables (or Frequency Distributions)

The frequency table is one of the important methods to present raw data in a form suitable for making the information contained in the raw data easily understandable.

We start with an example of a frequency table taken from the published results of the All India Agricultural Census conducted some years ago. We know that different persons own and cultivate different areas of land–some have more land than others, some have less. The Agricultural Census was carried out to obtain, besides other information, a fairly complete picture of the differences in size of land holdings in agriculture. For this purpose the Agricultural Census recorded the total land area which was in use wholly or partly for agricultural production and was operated as a single technical unit. Thus a holding of, say, one hectare in the Census records does not necessarily mean one piece of land (or a single field) measuring one hectare in area; it could also consist of more than one piece of land with the total area of all the pieces being one hectare. The raw data in this case would again be unintelligible but, presented in the form of Table 17.3,

TABLE 17.3

Number and Area of Holding by Size, Class of Holdings, Uttar Pradesh (Plains)

Sl. No.	*Size-class (in hectares)*	*Total No.*	*Holdings Area*		
1	*2*	*3*	*4*	*3(a)*	*4(a)*
1.	Below 0.5	7116591	1521670	46.9	8.6
2.	0.5-1.0	2986638	2143859	19.7	12.1
3.	1.0-2.0	2591431	3642575	17.1	20.5
4.	2.0-3.0	1089301	2632107	7.2	14.8
5.	3.0-4.0	533765	1829241	3.5	10.3
6.	4.0-5.0	300480	1333858	2.0	7.5
7.	5.0-10.0	428585	2856156	2.8	16.1
8.	10.0-20.0	94727	1230970	0.6	6.9
9.	20.0-30.0	11752	278311	0.1	1.6
10.	30.0-40.0	3198	108131	0.02	0.6
11.	40.0-50.0	1058	46710	0.007	0.3
12.	50.0 and above	1406	138150	0.009	0.8
	Total	15158932	17761738	99.936	100.1

1. Columns 1 to 4 have been taken from Table III-I, *Agricultural Census in Uttar Pradesh*, 1970-71, Board of Revenue, U.P., Lucknow, p. 126
2. Columns 3(a) and 4(a) are percentages obtained from columns 3 and 4, respectively

it presents a very clear picture about the amount of agricultural land held by different persons.

In the second column of Table 17.3 we have the values of the variable: 'amount of land' divided into 12 *classes* (or groups). In the third column we have the number of farmers who were cultivating land with total area falling in a given class. Thus, 7,116,591 farmers cultivated land which was less than 0.5 hectares in area. Next came 2,986,638 farmers whose cultivated land had area between 0.5 and 1 hectare. Columns 2 and 3 of this table constitute a *frequency table*. We ignore the other columns for the present.

There are two steps in drawing up a frequency table from raw data. The first step is to use the *possible values of a variable of observation* (land area) to define a number of classes. The second step is to count the number of units of observation (farmers) for which the values of the variable fall in a given class and to write down this number against the class. The number against each class is called the *frequency* of that class, and the total of all frequencies is called the *total frequency*. The total frequency is simply the number of *all* the units of observations for which we have recorded the value of the variable of observation.

A frequency table can then be described by giving the variable and the units of observation. The description is briefly conveyed by the title. For example, the frequency table formed by columns 2 and 3 of the table from the Agricultural Census would have

as its title: Frequency table *of* farmers *by* size of land cultivated.

The frequency table simply breaks up, or *distributes* the total frequency into the different classes of the table defined by means of the values of the variable of observation. For this reason we also use the term "frequency distribution" in place of "frequency table". The use of the word "table" draws attention to the form in which the raw data has been presented, whereas the word "distribution" emphasises that the total frequency has been divided into the frequencies of the different classes.

Care has to be taken while dividing the possible values of the variable into classes to form a frequency table to ensure that no possible values belongs to more than one class, and that every possible value belongs to some class or the other. If any value belongs to more than one class, the corresponding unit of observation would then be counted in the frequency of more than one class, so that the total frequency in the table would become *more* than the total number of units of observation. Again, if any value is left out, that is, does not belong to any class in the table, then a unit of observation having that value of the variable would not be counted in any of the class frequencies and the total frequency would be *less* than the total number of units of observation.

It should be kept in mind that the classes of a frequency table can consist of single values of the variable also; they need not always consist of more than one value.

If we look again at the frequency table of land holdings we find that the classes as stated appear to overlap with some values belonging to more than one class. Thus, the value 4.0 appears in the class 3.0 - 4.0, and also in the class 4.0 - 5.0. It would thus seem that the classes of the frequency table have not been defined correctly. However, a closer look at the table shows that this is not the case. The starting class has been difined as less than 0.5 hectares, and the class at the end is defined as '50 hectares *and above*'. We can thus conclude that the classes 0.5-1.0, 1.0-2.0 .... etc., really stand for "0.5 hectares and above but less than 1.0 hectares", "1.0 hectares and above but less than 2.0 hectares", and so on. With this explanation we find that the classes of the frequency table have been defined properly.

As another example we construct a frequency table of the population of Uttar Pradesh by age from the census table given earlier. The division into classes by age is already given in it, the different classes being 0-9, 10-14, 15-19, ..., 65-69, 70 and above. Against each class we write the number of persons of that age from the column "persons" and in the row marked "T" the total population is 88,341,144. However, the ages of 7,261 persons are not known. Hence, we take the total frequency as 88,341,144–7,261 = 88,333,883 to get the frequency table (Table 17.4).

TABLE 17.4

Frequency Table of Inhabitants of Uttar Pradesh by Age (1971 Census)

Age	*Frequency*	*%*
0-9	26,105,403	29.5
10-14	10,859,933	12.3
15-19	7,184,548	8.1
20-24	6,531,468	7.4
25-29	6,474,870	7.3
30-34	5,910,572	6.7
35-39	5,174,221	5.9
40-44	4,737,880	5.4
45-49	3,676,085	4.2
50-54	3,598,058	4.1
55-59	2,103,947	2.4
60-64	2,636,013	3.0
65-69	1,241,876	1.4
70 and above	2,099,009	2.4
Total	88,333,883	

The first two columns constitute the frequency table. In the third column we have expressed the frequency of each class as a percentage of the total frequency. The usefulness of this column will be discussed later.

In this table too the division into classes does not seem to have been properly done. Though the classes do not overlap, that is, any two classes do not have a common value, yet some *possible* values of the variable are not included in any class. For example, the beginning class 0-9 ends at 9 and the next class starts with the value 10. Thus the ages between 9 and 10 years are in neither of the two classes. However, we know that in census records the age of a person is recorded in *completed* years, so that, persons who have completed 9 years of age but are still below 10 years in age will be shown to have age 9 years. The class 0-9 therefore includes all people below 10 years of age and really stands for "Below 10 years". Similarly, the class 10-14 stands for "10 years and above but below 15 years". With this interpretation the frequency table is seen to have been correctly made.

It should have been clear by now that one may come across cases where the classes of a frequency table appear to overlap, or exclude some possible values of the variable. But the classes are seen to be properly defined if a certain rule of interpretation of the values belonging to a class is followed. The rule is either stated clearly or, is evident from the context, or from the manner in which the values of the variable are recorded.

We can obtain many more frequency tables for the census data table, such as

(i) frequency table of urban males by age,

(ii) frequency table of inhabitants by place of residence,

(iii) frequency table of rural inhabitants by sex.

Can you make these and other possible frequency tables yourself? Make note that while using sex or place of residence as the variable the total frequency will be the total number of persons in the census table. The number of persons in the category 'Age not stated' at the end of the table will not be subtracted in this case.

17.6 Construction of Frequency Tables from Raw Data

The construction of a frequency table from given raw data is most conveniently done by the use of the method of tally marks. Suppose we have the following raw data giving the ages of 50 persons.

43, 17, 27, 2, 19, 63, 84, 9, 11, 31,
22, 19, 44, 43, 6, 14, 71, 58, 39, 27,
1, 25, 36, 64, 42, 51, 40, 18, 19, 17,
63, 40, 26, 37, 11, 16, 21, 47, 68, 70,
21, 22, 46, 35, 23, 14, 50, 48, 61, 5

Let us divide the ages into the following classes: 0-9, 10-14, 15-19, 20-24, 25-29, 30-34, 35-39, 40-44, 45-49, 50-54, 55-59, 60-64, 65-69, 70-(i.c., 70 or more).

The first column of our frequency table will show these groups. We now look at the first value recorded in our raw data. It is 43 and falls in the class 40-44. In the row corresponding to this class we make a mark '|' in the second column of the table entitled 'tally marks'. The second value in the raw data is 11 and for it we make the mark '|' in the row corresponding to the class 10-14. By the time we have covered the first 26 values of the raw data, ending at 51, there will be four tally marks for the class 40-44 corresponding to the values 43, 44, 43, 42. These four tally marks will appear as '||||' in our table. The 27th value in the raw data is 40 and it also gives a tally mark in the class 40-44. This fifth tally mark will be entered as '$\cancel{||||}$' by crossing diagonally the four tally marks already entered there. Thus every fifth tally mark in a class is entered by crossing diagonally the four tally marks preceding it. If there are 13 tally marks in all in a class they will appear as '$\cancel{||||}$ $\cancel{||||}$ |||'. This method of recording tally marks makes counting the total number easier. The total number of tally marks of each class is then entered in the third column and gives the frequency of that class. Thus, our frequency table will finally look like this:

TABLE 17.5

Variable	*Tally Marks*	*Frequency*
0-9	𝍸	5
10-14	||||	4
15-19	𝍸 ||	7
20-24	𝍸	5
25-29	||||	4
30-34	|	1
35-39	||||	4
40-44	𝍸 |	6
45-49	|||	3
50-54	||	2
55-59	|	1
60-64	||||	4
65-69	|	1
70-	|||	3
Total		50

17.7 Relative Frequency Table

As we have explained earlier the frequency of any class in a frequency table is the number of units of observation for which the values of the variable belong to that class. Sometimes, this number (i.e. frequency) is expressed as a fraction of the total frequency and the fraction so obtained, usually expressed as a percentage, is called the *relative frequency* of the class. The relative frequencies in a frequency table make it easier to understand and assimilate the information, particularly, when the class frequencies are large.

For example, if we look at the Frequency Table 17.4 in which the numbers of persons belonging to different age-groups are given, the individual class frequencies are so large that we cannot make comparisons between different age groups easily. If, however, we look at the relative frequencies we see that almost a third (29.5%) of the population is below 10 years in age. We also notice that persons of age 55 years or more constitute only about one-tenth $(2.4 + 3.0 + 1.4 + 2.4 = 9.2\%)$ of the total population. It is also clear that the number of persons of age 35 or more (28.8%) is nearly equal to the number of children of age less than 10 years, and this fact cannot be noticed so easily if one looks at the actual frequencies.

Similarly, if we look at the relative frequencies in the frequency table of land-holding given in column 3(a) of Table 17.3 certain interesting features stand out much more clearly as compared to the table of actual frequencies. For example, though there are persons owning more than 50 hectares of land their number is very small, just about .01%. Majority of persons own small pieces of land; nearly half (46.9%) own less than half a hectare of land, and two-thirds $(46.9 + 19.7 = 66.6\%)$ own less than one hectare of land.

You may have noticed that the total of relative frequencies, which should be exactly 100 in theory, is more than 100 (=100.1) in the census example, and less than 100(=99.936) in the land-holdings example. This is really not as surprising as it may appear at first; the difference from the theoretical value of 100 arises due to the fact that percentages given there are not exact but close approximations of the exact value. In the census frequency table the percentage 29.5 stands for

$$\frac{26105403}{88333883} \times 100 = 29.553\ldots$$

and is thus *less* than the true value. Similarly, for the age-group 40-44 the relative frequency 5.4% stands for

$$\frac{4737880}{88333883} \times 100 = 5.363$$

and is therefore, *more* than the true value. While calculating the percentages we have *rounded off* the percentage due to which some percentages are more and some less than the exact percentage so that the total of all percentages will sometimes be more, and sometimes less than 100. Can this total be sometimes equal to 100 also?

Relative frequencies are also very useful when we have to compare two or more frequency distributions. For example, suppose the ages of the inhabitants of two villages result in the following frequency distributions, where the relative frequencies are also shown.

TABLE 17.6

	No. of inhabitants		*Relative frequency (%)*	
Age-group	*Village A*	*Village B*	*Village A*	*Village B*
0-5	18	35	12.6	15.3
5-10	26	38	18.2	16.7
10-20	25	40	17.5	17.5
20-30	29	32	20.3	14.0
30-40	21	27	14.7	11.8
40-50	14	23	9.8	10.1
50-60	6	18	4.2	7.9
60-80	4	15	2.8	6.6
Total	143	228	100.1	99.9

The relative frequencies express the frequency of any class as a percentage of the total frequency. Thus for village A, the relative frequency of the 0-5 class is $\frac{18}{143} \times 100 = 12.6$, while for village B it is $\frac{35}{228} \times 100 = 15.3$.

When we look at the frequencies of the different age-groups of the two villages, we cannot easily see if the two are similar or different. We cannot, for example, find out easily if one village consists mostly of younger or older people as compared to the other. When we examine the relative frequencies, we can quickly discern some interesting features. Village B has a higher proportion of children in age-group 0-5. If we look at age-groups 5-10 and 10-20 taken together, we find that the proportion of inhabitants of ages from 5 to 20 years ($18.2 + 17.5 = 35.7\%$) in village A is almost the same as the proportion of such persons ($16.7 + 17.5 = 34.2\%$) in village B. As we go down the relative frequency table, we see that village A has a higher proportion of persons with ages from 20 to 40 years than village B. The proportions for them being 35.0% (20.3 + 14.7) and 25.8% (14.0 + 11.8) respectively. Consequently, the proportion of older persons with age ranging from 40 to 80 years is more in village B ($10.1 + 7.9 + 6.6 = 24.6\%$) than in village A ($9.8 + 4.2 + 2.8 = 16.8\%$).

It must be noted that for comparing two frequency distributions by means of the relative frequencies, the variable as well as the division of the values of the variable into classes, must be the same for both distributions.

17.8 Graphical Presentation of Frequency Distributions

The main features of a frequency distribution are conveniently communicated by representing the frequency distribution in the form of a diagram, since a diagram is more easily and more quickly understood than a collection of numbers. Diagramatic presentation is particularly useful when the number of classes or the class frequencies in a frequency distribution is large.

There are various methods of graphical presentation of frequency distributions which are in use. We shall only discuss two of them, viz., the *bar diagram* and the *pie diagram*.

Consider the example of the frequency distribution by age of inhabitants of village A which we introduced while discussing the use of the concept of relative frequency. The *bar diagram* for this frequency distribution is shown in Fig. 17.1.

To draw the bar diagram we mark equal lengths for the different classes on the horizontal axis. Notice that the same length is used for every class even though different classes have different age ranges. Some span a period of 5 years, some 10, and the last one spans a period of 20 years. On each of these lengths on the horizontal axis we erect a rectangle whose height is *proportional* to the frequency of the class represented by the base. We thus get a number of "bars" which give the diagram its name. We could also make the heights of the rectangles proportional to the relative frequencies to get the bar diagram of the relative frequency distribution. In fact, only a single bar diagram has to be drawn which serves both for the frequency distribution as well as the relative frequency distribution. The scale on the vertical axis indicates whether the bar diagram represents the frequencies or the relative frequencies. In Fig. 17.1, we have shown the scale for the frequencies on the left, and for the relative frequencies on the right.

If we have two frequency distributions with the same classes we can represent both

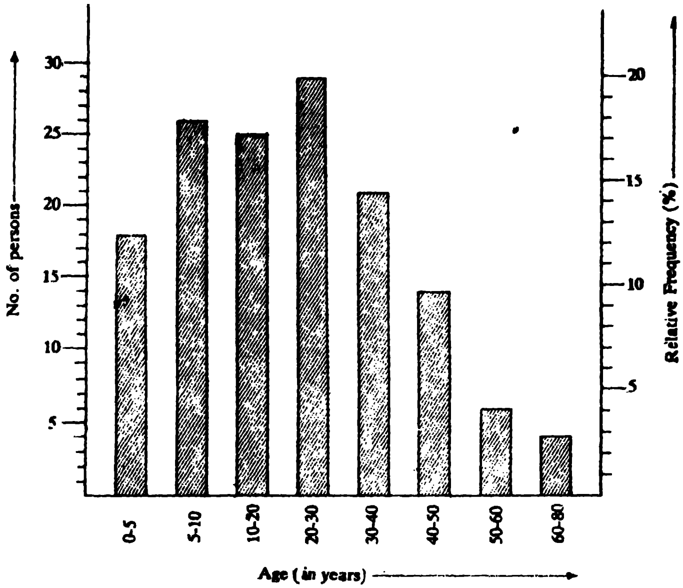


Fig 17.1

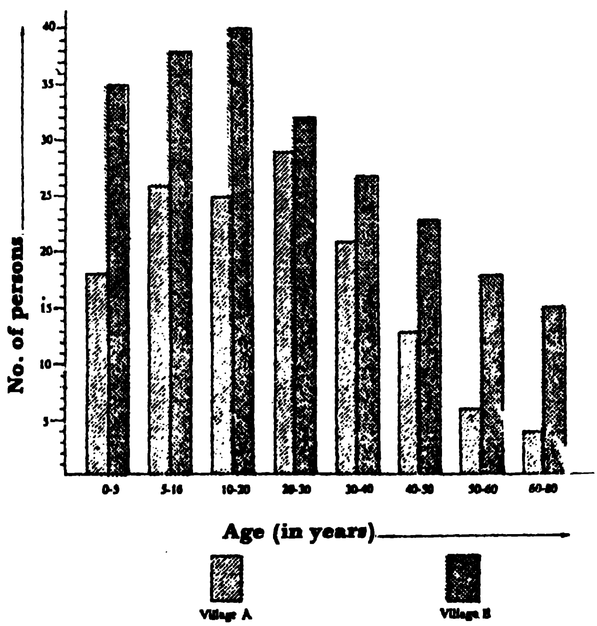


Fig 17.2

on the same bar diagram. For example, if we have the distributions by age of village A and village$B$ in the example mentioned above, we can represent the two frequency distributions in the same diagram as shown in Fig. 17.2. The vertical scale for the frequencies is shown on the left.

However, we cannot represent the two frequency distributions as well as their relative frequency distributions in a single diagram as we were able to do for a single frequency distribution. If we are interested in comparing graphically two frequency distributions with the same classes, we have to represent their relative frequency distributions in a single diagram. This has been done for our example in Fig. 17.3 in which the vertical scale now represents the relative frequency percentages.

In the *pie diagram*, which should be used for representing relative frequency distributions only, the relative frequencies are represented by *sectors* of a circle. We start with a circle of arbitrary radius and then draw radii from the centre to the circumference. In this way the circle is divided into as many parts as there are classes in a frequency distribution. The *area* of each sector is proportional to the relative frequency of the class represented by the sector.

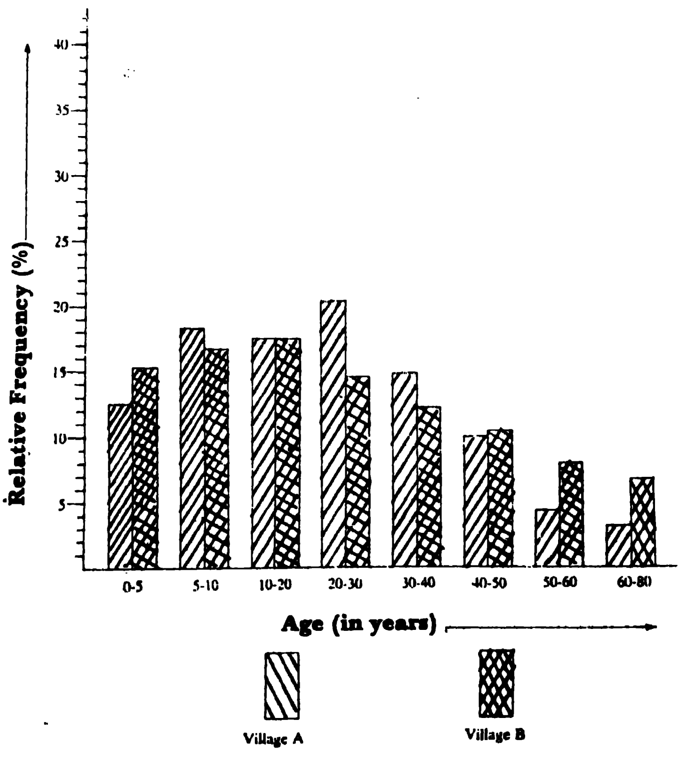


Fig 17.3

We take again the example of village A. There are eight classes, and the relative frequencies (expressed as percentages) are 12.6, 18.2, 17.5, 20.3, 14.7, 9.8, 4.2 and 2.8. The *pie diagram* for this relative frequency distribution is shown below:

We start by drawing a radius OA. This is followed by drawing the radii OB, $OC, \ldots$ so that the area of the sector AOB is proportional to 12.6, that of sector BOC is proportional to 18.2, and so on, till all the radii are drawn to divide the area of the circle into eight sectors. We know from our knowledge of geometry of the circle that the area of any sector is proportional to the angle subtended at the centre by its arc. Hence, what we require is that the angles AOB, BOC be proportional to 12.6, 18.2, ... respectively. Since, the total of all the eight angles at the centre is 360°, the eight angles are determined by dividing 360° into eight parts in the proportion 12.6 : 18.2 : 17.5: ... 4.2 : 2.8. Hence,

$$\angle AOB = \frac{12.6 \times 360}{100.1} = 45.3^\circ$$

$$\angle BOC = \frac{18.2 \times 360}{100.1} = 65.4^\circ$$

$$\angle COD = 62.9^\circ$$
$$\angle DOE = 73.0^\circ$$
$$\angle EOF = 52.8^\circ$$
$$\angle FOG = 35.2^\circ$$
$$\angle GOH = 15.1^\circ$$
$$\angle HOA = 10.0^\circ$$

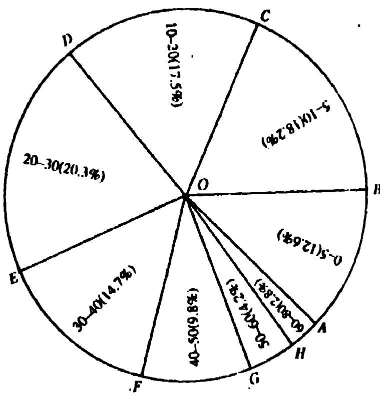


Fig. 17.4 Distribution of population of village A in various age-groups

The pie diagram can be easily drawn when these angles have been obtained.

17.9 Measures of Location and Dispersion

So far we have discussed the presentation of raw data in a form suitable for communicating the information contained in it, and have studied the use of the frequency table for this purpose.

In the case of *quantitative variables* the information contained in the raw data, or in the associated frequency table, can also be *summarised* by means of a few numerical values. Such a summary is partly provided by what are called *measures of location* (also called *measures of central tendency*) and *measures of dispersion*.

17.10 Measures of Location

The two most commonly used measures of location are the *arithmetic mean* (or the mean in short) and the *median*. The mean and the median lie between the largest

and the smallest of the observations, and are, therefore, also called measures of central tendency. Both can be said to lie, in some sense, at the centre of the observations. But the precise meaning of the centre is, as we shall see, different for the mean and the median.

(a) *The Arithmetic Mean*

The arithmetic mean of a given set of observations is simply the *average* of all the values of the variable recorded in the raw data. The raw data from which we constructed the frequency table by the method of tally marks consisted of the ages of 50 persons. The mean of these 50 values is given by

$$\frac{43 + 17 + 27 + \ldots + 48 + 61 + 5}{50} = \frac{1686}{50} = 33.72$$

We say that the *mean age* of the 50 persons is 33.72 *years per person*. Note that we do not simply say that the mean age (or mean) is 33.72 but also mention the unit of measurement in which the quantitative variable has been measured (here it is years) as well as the units of observation (here persons) from whom the observations have come. In the same way, if the average marks of a class of students equal 47.5, we will say that the mean is 47.5 marks per student.

The general definition of the arithmetic mean is as follows:

Definition: The arithmetic mean of the values $x_1, x_2, \ldots, x_n$ of a variable recorded for n units of observation is defined as $\frac{x_1 + \ldots + x_n}{n}$ per unit of observation.

The expression $x_1 + \ldots + x_n$ is usually denoted by $\sum_{i=1}^{n} x_i$ (or simply $\sum x_i$), and the mean, denoted usually by $\bar{x}$. Therefore, $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$.

The mean provides a summary of the information contained in the raw data $x_1, \ldots, x_n$. Being a summary it does not communicate all the information in the raw data. If we know $\bar{x}$, we cannot say anything about the actual values $x_1, x_2, \ldots, x_n$. But, if with $\bar{x}$ we also know the value of n, we can obtain the total of all the values $x_1, \ldots, x_n$, which from the definition of $\bar{x}$ can be seen to be equal to $n\bar{x}$. We also know that some values are more and some less than $\bar{x}$. Thus, in some sense, $\bar{x}$ lies at the centre of all the values.

From the definition of $\bar{x}$ we get

$$\sum_{i=1}^{n}(x_i - \bar{x}) = 0 \quad \text{i.e.}$$

$$(x_1 - \bar{x}) + (x_2 - \bar{x}) + \ldots + (x_n - \bar{x}) = 0.$$

Thus, some values $x_i - \bar{x}$ must be positive and some negative, otherwise the sum would not be zero.

If we add all the positive $x_i - \bar{x}$ and all the negative $x_i - \bar{x}$, these two sums will have the same *magnitude* but will have different signs, so that their algebraic sum is zero. When we say the $\bar{x}$ lies at the centre of all the observations, we are simply saying that the sums of the positive $x_i - \bar{x}$ and the negative $x_i - \bar{x}$ have the same magnitude, and the algebraic sum is zero. In this case being at the centre does not imply that half the observations are more than $\bar{x}$ and half are less than it. For, it can happen that $x_i - \bar{x}$ is positive for a small number of observations and negative for a large number, or vice-versa.

Calculation of Arithmetic Mean from Frequency Tables

Suppose we have the following frequency table of families by the number of children in which each class of the frequency table is defined by a *single* value of the variable of observation.

TABLE 17.7

No. of children	*Frequency*
0	43
1	55
2	60
3	64
4	48
5	34
Total	304

The table tells us that the number of children in each of the 304 families being studied was recorded, and it was found that 43 families had no children, 55 had only 1 child, and so on. There are thus 304 values of the variable which have been observed and the value 1 occurs 55 times, 2 occurs 60 times, and so on. To get the mean of these 304 values, we have first to add all of them. That sum is most conveniently obtained by using multiplication and is given by

$$(0 \times 43) + (1 \times 55) + (2 \times 60) + (3 \times 64) + (4 \times 48) + (5 \times 34) = 729$$

Thus the mean is $\frac{729}{304} = 2.4$ (approx.) children per family.

In general,if the values in the different classes of the frequency table are $x_1, \ldots, x_m$ and the frequencies of $f_1, \ldots, f_m$, the mean will be given by

$$\bar{x} = \frac{\sum_{i=1}^{m} f_i x_i}{\sum_{i=1}^{m} f_i}.$$

TABLE 17.8

Variable	*Frequency* f	*Mid-value* x	$f.x$
Below 0.5	7,116,591	0.25	1,779,147.75
0.5 - 1.0	2,986,638	0.75	2,239,978.50
1.0 - 2.0	2,591,431	1.50	3,887,146.50
2.0 - 3.0	1,089,301	2.50	2,723,252.50
3.0 - 4.0	533,765	3.50	1,868,177.50
4.0 - 5.0	300,480	4.50	1,352,160.00
5.0 - 10.0	428,585	7.50	3,214,387.50
10.0 - 20.0	94,727	15.00	1,420,905.00
20.0 - 30.0	11,752	25.00	293,800.00
30.0 - 40.0	3,198	35.00	111,930.00
40.0 - 50.0	1,058	45.00	47,610.00
Total	15,157,526		18,938,495.25

Consider next the frequency table of farmers by the area of land held, given earlier. Here the classes are no longer defined by single values; each class consists of many values of the variable. In such a case, we cannot determine the *exact* value of the mean because all the values of the variable are not known exactly. So, what one tries to do is to get an *approximate* value of the mean. The method of approximation is to replace all the observed values belonging to a class by the *mid-value* of that class and then use the mid-value to determine the mean. In this particular example we are faced with a difficulty as the last class is defined as "50 hectares and above" so that its mid-value cannot be obtained. So we ignore this class, and use the remaining classes only to illustrate the method.

The calculations are presented as follows by adding two more columns to the frequency table for the mid-value and the product of the mid-value and the frequency.

What we have done is to treat each class as defined by a single value, the mid-value of that class, and then followed the method applicable to such frequency tables.
The approximate value of the mean is given by

$$\begin{aligned}\bar{x} &= \frac{18,938,495.25}{15,157,526} \text{ hectares per cultivating unit.}\\ &= 1.249 \text{ hectares per cultivating unit.}\\ &= 1.25 \text{ hectares (approx.) per cultivating unit.}\end{aligned}$$

EXERCISE 17.1

1. Prove that $(x_1 - \bar{x}) + (x_2 - \bar{x}) + \ldots + (x_n - \bar{x}) = 0$
   where $\bar{x} = \dfrac{x_1 + x_2 + \ldots + x_n}{n}$

2. (a) The marks obtained by 20 students in a test were
13, 17, 11, 5, 18, 16, 11, 14, 13, 12, 18, 11, 9, 6, 8, 17, 21, 22, 7, 6
Find the mean marks per student.
(b) If extra 5 marks are given to each student, show that the mean marks are also increased by 5.
(c) If 2 marks are deducted for each student, show that the mean marks are also reduced by 2. (See question 3 for the reason).

3. If $\bar{x}$ is the mean of $x_1, x_2, \ldots, x_n$, show that the mean of $x_1 + a,\ x_2 + a, \ldots, x_n + a$ is $\bar{x} + a$, where a is any number, positive or negative.

4. (a) *If all the marks in 2(a) are doubled, show that the mean value is also doubled.*
(b) The postal expenses on the letters despatched from an office on a given day resulted in the following frequency distribution:

Postage (p)	*No. of letters*
15	47
30	33
35	56
60	41
70	25

Find the mean postage per letter. Convert the postal charges in rupees and then calculate the mean postage per letter.

5. If $\bar{x}$ is the mean of $x_1, x_2, \ldots, x_n$, show that the mean of $ax_1, ax_2, ax_3 \ldots, ax_n$, is $a\bar{x}$, where a is any number different from zero.

6. The mean age (in years) per student and the number of students in each of the four classes of two primary schools are given below:

	School A		*School B*	
	No.	*Mean age*	*No.*	*Mean age*
Class I	6	6.2	25	7.1
Class II	10	7.5	32	8.4
Class III	28	8.6	12	9.2
Class IV	30	10.0	4	10.7

(a) Obtain the mean age per student for the two schools.
(b) In each class, the mean age of students of school A is less than that of students in the same class of school B. But we find that the mean age per student for the whole school is more for school A than for school B. Why?

7. The ages of all the male inhabitants of a village were recorded and the following frequency table was obtained:

Age (Years)	*No. of persons*
0–5	12
5–10	18
10–20	16
20–30	19
30–40	14
40–50	11
50–60	4
60–80	3

Obtain the mean age per male inhabitant.

8. The measurements (in mm) of the diameters of the heads of 107 screws gave the following frequency table:

Diameter	*Frequency*
33–35	17
36–38	19
39–41	23
42–44	21
45–47	27

Calculate the mean head diameter per screw.

(b) *The Median*

The median is also a value lying at the centre of all the observations but in a different sense as compared to the mean. In the case of the median the number of values which are more than the median equals the number of values which are less than it. For example, suppose the values of the variable are

15, 5, 7, 3, 11, 28, 9, 43, 28, 17, 31

Then five values (3,5,7,9,11) are less than 15, and five values (17, 28, 28, 31, 43) are more than it. Thus we can say that 15 is the *median* (or *median value*) of the given set of values.

In the above example, the number of values was 11, an odd number, so that one could obtain 5 values more and 5 less than the value at the centre. Let us see what happens when the total number of observations is an even number. For example, let the values be

15, 5, 7, 3, 11, 16, 28, 9, 43, 28, 17, 31.

Suppose we count for each value, starting with the smallest, the number of values which are more than it and the number of values which are less. We obtain the following table.

TABLE 17.9

Value (x)	3	5	7	9	11	15	16	17	28	31	43
No. of values $< x$	0	1	2	3	4	5	6	7	8	10	11
No. of values $> x$	11	10	9	8	7	6	5	4	2	1	0

In this case, we do not find any value which is such that the number of values more than it *equals* the number of values less than it. One could however agree to call the values 15 and 16 as coming nearest to the definition of median. Also, if we take the number 15.4, which is not an observed value, it could be called the median as it has exactly 6 values more and 6 less than it. But then *any* number between 15 and 16, say 15.6 or 15.7, could also serve as the median of the observations. Looked at in this way one finds that the median may not be a single value or number, but can be one of a set of values.

Another problem is created when a large number of values are repeated. Consider the eleven observations (arranged in ascending order)

3, 5, 7, 8, 15, 15, 15, 15, 15, 15, 43.

We have 11 observations and we cannot pick any value, or even any number as the median in the sense that it will have equal number of values above and below it.

The general idea of the median as a measure of central tendency is easy to understand. We are looking for a value, or a number, which divides the given observed values into two parts, half the number of observations being more, and half less, than the median value. The median is not only a value lying at the centre of the observations but also conveys very useful information. Thus, if we say that the median monthly income of a group of workers is Rs. 500.00 then it is clear that half the workers earn more, and half less than Rs. 500.00 per month. But its calculation poses some difficulties as we have seen from the examples considered by us. These difficulties are overcome by a slight modification in the definition which is as follows:

Definition: The median of a given set of n values of a quantitative variable is *any number* M such that

(i) [No. of values $\leq M$] $\geq \frac{n}{2}$ (ii) [No. of values $\geq M$] $\geq \frac{n}{2}$

Let us apply this definition to the three examples considered above.

In the first example, the values when arranged in increasing order are

3, 5, 7, 9, 11, 15, 17, 28, 28, 31, 43.

Here $n = 11$, $\frac{n}{2} = 5.5$. For a number M to be the median, we require

i) (No. of observations $\geq M$) ≥ 6

(ii) (No. of observations $\leq M$) ≥ 6

To satisfy (i), we count 6 observations from the right and see that we must have $M \leq 15$. Similarly, counting 6 observations from the left, we see that we must have $M \geq 15$. Hence, $M = 15$.

In the second example, the observations arranged in increasing order are

$$3, 5, 7, 9, 11, 15, 16, 17, 28, 28, 31, 43.$$

Here $n = 12$, $\frac{n}{2} = 6$. Hence, counting 6 observations from the right, we see that $M \leq 16$. Similarly, counting 6 observations from the left, we see that $M \geq 15$. Thus 15 and 16 and any number lying between them can be taken as the median according to our definition.

Applying the same procedure to the third example, we find that 15 is the median for the set of observations

$$3, 5, 7, 8, 15, 15, 15, 15, 15, 43.$$

Here $\frac{n}{2} = 5$, and the 5th observations from the right and from the left are both equal to 15. Hence, $M = 15$.

The use of the general definition determines the median without ambiguity. However, in some cases the median may not be unique. In such cases we follow a *convention* (that is, a procedure or rule accepted by everyone) according to which the average of the smallest and largest median values is taken as the median. In the second example, the smallest median value was 15 and the largest median value was 16. So, by our convention we take 15.5 as the median in this case.

Calculation of the Median: If the observations are given as raw data, we arrange them in increasing (or decreasing) order and count the total number of observations. If the number n of observations is odd, say, $n = 2m + 1$, then $\frac{n}{2} = m + \frac{1}{2}$. Hence, from the definition of median, we find, as in the first example above, that the $(m+1)$th observation is the median. For example, if there are 13 observations, we have $13 = 2 \times 6 + 1$, so that $m = 6$. Hence, in this case the 7th observation will be the median.

If n is even, say, $n = 2m$, we have $\frac{n}{2} = m$. Then, as in the second example above, the median will be any number between the mth observation from the right and the mth observation from the left. The average of these two observations is then taken as the median. In other words, we take the average of the mth and the $(m + 1)$th value as the median.

If the observations are given in the form of a frequency table with classes defined through single values, we rearrange the frequency table so that the values of the variable appear in ascending or descending order. We then add a column of *cumulative frequencies*, the cumulative frequency of any class being the total of the frequencies of that class and all classes coming before it in the table. If the total frequencies is odd $(= 2m + 1)$, we locate, by means of the cumulative frequencies, the class in which the $(m + 1)$th observation falls and the value defining that class is the median. If the total frequency

is even ($= 2m$), we locate similarly the classes and then the values corresponding to the mth and $(m+1)$th observations and take their average as the median.

As an example, we consider the frequency table of families by number of children, which we used to illustrate the method of calculating the arithmetic mean (see Table 17.7). The total frequency is 304, an even number. As $304 = 2 \times 152$, we have to locate the 152nd and 153rd observations. For that we look at the column of cumulative frequencies in the table given below:

TABLE 17.10

No. of Children	*Frequency*	*Cumulative Frequency*
0	43	43
1	55	98
2	60	158
3	64	222
4	48	270
5	34	304
Total	304	

From the column of cumulative frequencies we notice that the first 43 observations are all equal to 0, all the observations from the 44th to the 98th are equal to 1, all from the 99th to the 158th are equal to 2 and so on. Thus, the 152nd and 153rd observations are both equal to 2 so that the median number of children per family is 2. What it suggests is that of the families observed half had less and half more than 2 children.

Consider now the case where each class of the frequency table consists of more than one value. As in the case of the mean, we cannot now calculate the exact value of the median and have to go in for an approximation. The approximation method, however, is different from that which was adopted for the calculation of the mean.

We first arrange the classes of the frequency table in ascending or descending order of the values of the variable. Then we locate, with the help of the cumulative frequencies, the class in which the median value lies. However, since we are calculating an approximate value only, we make no distinction between an odd or an even number of observations. If n is the total number of observations, we obtain the value of $\frac{n}{2}$ and then locate the class in which the median lies as the class for which the cumulative frequency first equals or exceeds the value $\frac{n}{2}$.

Having located the class in which the median lies, called the *median class*, we now use our approximation method to find the actual median value in that class. For approximation we assume that all the observations of that class are uniformly spread over the whole class. Thus, if the class consists of values between 15 and 25 and there are 20 observations falling in that class, we *assume* that these 20 observations are so spread that the difference between two consecutive observations is $\frac{25-15}{20} = 0.5$. Thus, the 20 observations can be taken as 15.5, 16.0, 16.5, 17.0, 17.5, 18.0, 18.5, 19.0, 19.5,

TABLE 17.11

Variable	Frequency	Cumulative Frequency
Below 0.5	7,116,591	7,116,591
0.5 - 1.0	2,986,638	10,103,229
1.0 - 2.0	2,591,431	12,694,660
2.0 - 3.0	1,089,301	13,783,961
3.0 - 4.0	533,765	14,317,726
4.0 - 5.0	300,480	14,618,206
5.0 - 10.0	428,585	15,046,791
10.0 - 20.0	94,727	15,141,518
20.0 - 30.0	11,752	15,153,270
30.0 - 40.0	3,198	15,156,468
40.0 - 50.0	1,058	15,157,526
50.0 or more	1,406	15,158,932
Total	15,158,932	

20.0, ..., 24.5, 25.0. The median value inside the median class is then determined on this assumption.

As an example, consider the frequency table of farmers by the amount of land owned which we have already come across. To the column of frequencies we add the column of cumulative frequencies. No rearrangement of the classes is needed as they are already arranged according to ascending values of the variable. Notice that we have not excluded the class "50 hectares and above" which we had done while illustrating the calculation of the arithmetic mean, because now our method of approximation is not based on the use of mid-values of the classes.

Here $n = 15,158,932$, so the $\frac{n}{2} = 7,579,466$ which is less than the cumulative frequency of the "0.5 – 1.0" class and more than the cumulative frequency of the "Below 0.5" class. Hence, the median class is the "0.5 – 1.0" class and the median value lies between 0.5 and 1.0 hectare. To find the median, we have to locate the 7,579,466th observation in this class. The 7,116,591th observation can be taken to be 0.5 hectares. The next 2,986,638 observations equally spaced between 0.5 and 1.0, that is they cover a range of $1.0 - 0.5 = 0.5$ hectares. To reach the 7,579,466th observation, we have to find the 7,579,466 – 7,116,591 = 462875th observation in the median class. The first 462875 observations in the median class will, according to our assumption to get an approximate value of the median, cover a range of

$$\frac{0.5}{2986638} \times 462875 = 0.08 \text{ hectares.}$$

Hence, the median value is given by

$$0.5 + 0.08 = 0.58 \text{ hectares.}$$

We thus see that even though there are farmers owning 10,20, or even more than 50 hectares of land, nearly half the farmers own less than 0.58 hectares of land.

The general expression for finding the median in such cases is given by

$$M = l + \frac{\frac{n}{2} - f_0}{f_1} . h$$

where
M is the median,
l is the lower value of the median class,
f_0 is the cumulative frequency of the class just before the median class,
f_1 is the frequency of the median class,
h is the range (upper value − lower value) of the median class,
n is the total frequency.
It is assumed that the classes have been arranged according to ascending values of the variable.

EXERCISE 17.2

1. The number of students absent in a school was recorded every day for 147 days and the raw data was presented in the form of the frequency table below:

No. of Students absent	*No. of days*
5	1
6	5
7	11
8	14
9	16
10	13
11	10
12	70
13	4
15	1
18	1
20	1
Total	147

Obtain the median and describe what information it conveys.

2. The marks obtained out of 50 by 102 students in a test were according to the frequency table below:

Marks	*Frequency*
20	8
22	15
23	28
24	27
26	20
31	2
38	1
43	1

Obtain the median and describe what information it conveys.

3. The following table gives the frequency distribution of married women by age at marriage:

Age (in years)	*Frequency*
15 - 19	53
20 - 24	140
25 - 29	98
30 - 34	32
35 - 39	12
40 - 44	9
45 - 49	5
50 - 54	3
55 - 59	3
60 and above	2

Calculate the median and interpret the result.

4. In the example of land holdings we found that the mean was 1.25 hectares and the median only 0.58 hectares. What do you conclude from it?

17.11 Measures of Dispersion

Supposing you were the captain of a cricket team and had selected the whole team except for one more batsman. Two batsmen, whom we shall call A and B, were available and you had to choose one of them to complete the team. The runs scored by these two batsmen in the last 5 matches (10 innings) were as follows:

A: 38,70; 48,34; 42,55; 63,46; 54,44 (Total: 494)
B: 5,11; 8,29; 83,104; 20,28; 81,123 (Total: 492)

The average score per innings is nearly 50 for both the batsmen. So which one do you select? If your team already has batsmen on whom you can depend, you could select B in the hope that he will hit up a big score. Of course, you would then be taking

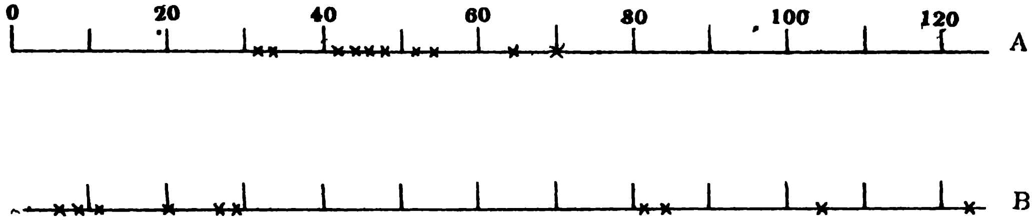


Fig. 17.5

the risk, that he may get dismissed on a poor score. If you do not want to take such a risk you would select A since his scores do not change much from innings to innings and you can be more certain that he would make a reasonably good score. Though the two batsmen have almost the same average score per innings, A's scores do not show much variation from innings to innings while B's scores show great variation with very high scores in some innings and very low scores in some others.

If we plot the scores of A and B on a straight line, we see that the points corresponding to the scores of A are close to each other, or bunched together, and those corresponding to the scores of B are scattered or spread out.

We use the term dispersion to indicate this scattering or spreading out of the different values of a quantitative variable. Thus, we will say that the scores of B show a higher dispersion than the scores of A.

Let us again look at the scores of A and B. The mean score per innings of A is 49.4 and his scores are close to the mean score, the difference between the scores and the mean score ranges from $34 - 49.4 = -15.4$ to $70 - 49.4 = 20.6$. The mean score for B is 49.2, but his scores in the different innings are not close to the mean value but located away from it as compared to A. In his case the difference between the scores and the mean value ranges from $5 - 49.2 = -44.2$ to $123 - 49.2 = 73.8$. We say that A's scores have a smaller *dispersion about the mean* as compared to B's scores.

We have already discussed the use of measures of location to *summarise* the information provided by the values of a quantitative variable. The measures of location, which we have studied, the mean and the median, give us a sort of central value around which the values of the variable are located, some values being more and some less than the central value. But they do not give us any idea of how far these values are from the central value. The measures of dispersion are designed to provide this information, that is to indicate if the different values of the variable are close to a measure of location or away from it. A *measure of dispersion about the mean* will tell us if the observations are close to the mean value or not. Similarly, a *measure of dispersion about the median* will tell us about the degree of scatter of the observations about the median.

The commonly used measure of dispersion about the mean is the *variance* or, the *standard deviation* which is simply the square root of the variance. The *measure of*

dispersion about the median, which we shall study, is the *mean deviation.*

A small value of the variance will indicate that most of the observations do not differ very much from the mean, and variance will go on increasing in value as the observations get further and further away from the mean value. Similarly, small values of the mean deviation will indicate that the observations are close to the median whereas large values will indicate that the observations differ much from the median.

The two measures of location, the mean and the median, provide a very inadequate summary of the information conveyed by a set of observations, but the mean along with the variance (or standard deviation), and the median along with the mean deviation provide a summary which is adequate in most situations.

(a) *Variance and Standard Deviation*

Consider the scores of batsman A. If we calculate the differences of these scores with their mean value, we get $38 - 49.4 = -11.4, 70 - 49.4 = 20.6$, and so on. The ten differences are $-11.4, 20.6, -1.4, -15.4, -7.4, 5.6, 13.6, -3.4, 4.6, -5.4$. Similarly, calculations for B give the ten differences equal to $-44.2, -38.2, -41.2, -20.2, 33.8, 54.8, -29.2, -21.2, 31.8, 73.8$. Comparison of *these differences show that they are on the whole* larger for B as compared to the differences for A. And because of that B's scores have a higher dispersion about the mean in comparison to A. We will see below that the definition of variance is based upon this idea.

Suppose our observations are $x_1, \ldots, x_n$ and their arithmetic mean is $\bar{x}$. Let us now look at the differences $x_1 - \bar{x}, x_2 - \bar{x}, \ldots, x_n - \bar{x}$. If these differences are small *in magnitude* it means that values $x_1, \ldots, x_n$ are close to the mean value $\bar{x}$, if they are large in magnitude it implies that the values are scattered away from the mean value. So, a measure of the degree of scatter around $\bar{x}$, that is a measure of dispersion around $\bar{x}$ could be constructed by utilising the differences $x_1 - \bar{x}, \ldots, x_n - \bar{x}$. If we simply add these differences, some of which will have a positive value and some a negative value, we will get the sum equal to zero. So, the sum of the differences cannot be used to obtain an idea of the total dispersion of the values about the mean $\bar{x}$. The difficulty is due to the fact that some differences are positive and some negative. We overcome this difficulty by *squaring* all the differences and then adding them to get the non-negative sum.

$$(x_1 - \bar{x})^2 + \ldots + (x_n - \bar{x})^2 = \sum_{i=1}^{n}(x_i - \bar{x})^2$$

If this sum is zero, each difference $x_i - \bar{x}$ must be zero so that there is no dispersion at all as all the observations are equal to the mean value $\bar{x}$. If this sum is small, all the individual differences will be small in magnitude, indicating that the values $x_1, \ldots, x_n$ are all close to the mean $\bar{x}$. If this sum is large then most of the individual differences will be large in magnitude (though some may be small) thus indicating a higher degree of dispersion about $\bar{x}$.

Thus the sum $\sum_{i=1}^{n}(x_i - \bar{x})^2$ is a reasonable indicator of the amount of dispersion about the mean value $\bar{x}$. But there is another difficulty to be overcome. It can happen that we have a large number of observations for which the individual differences $x_i - \bar{x}$ are small in magnitude so that we should get a small value for the measure of dispersion. But, it can happen that the sum $\sum_{i=1}^{n}(x_i - \bar{x})^2$ is larger in this case than when we have very few observations with comparatively larger values for the individual differences. We would then be led to conclude that the dispersion is more in the first case than in the second even though the situation is quite the opposite.

For example, the 5 numbers 10, 20, 30, 40, 50 have mean equal to 30 so that $\sum(x_i - \bar{x})^2$ in this case is equal to

$$(-20)^2 + (-10)^2 + (10)^2 + (20)^2 = 1000$$

And the 41 numbers
20, 20.5, 21, 21.5, 22, 22.5, 23, 23.5, 24, 24.5, 25, 25.5,
26, 26.5, 27, 27.5, 28, 28.5, 29, 29.5, 30, 30.5, 31, 31.5,
32, 32.5, 33, 33.5, 34, 34.5, 35, 35.5, 36, 36.5, 37, 37.5,
38, 38.5, 39, 39.5, 40
also have mean equal to 30 with $\sum(x_i - \bar{x})^2$ equal to

$$(-10)^2 + (-9.5)^2 + (-9)^2 + \ldots + (-0.5)^2 + (0.5)^2 + \ldots + (9.5)^2 + (10)^2 = 1435.$$

Thus, if we use $\sum(x_i - \bar{x})^2$ as our measure of dispersion about the mean, we would say that the second set of numbers has a higher dispersion about the mean than the first. Obviously, such a conclusion would be wrong, as all the individual differences $x_i - \bar{x}$ lie between -10 and 10 in the second case while in the first the individual differences $x_i - \bar{x}$ range from -20 to 20. So, in effect the observations in the first case are not as close to the mean as in the second and should, therefore, give a higher value for the dispersion.

The difficulty is created by the fact that the total number of observations in the two cases is different. In the second case, though the values $x_i - \bar{x}$ are small, the total $\sum(x_i - \bar{x})^2$ for 41 observations becomes larger than the similar total for the first case in which there are only 5 observations.

We overcome this difficulty by dividing the total $\sum(x_i - \bar{x})^2$ by the number of observations. The resulting value $\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$ is called the *variance* of the set of observations $x_1, x_2, \ldots, x_n$. With this modification, the dispersion of the first set of observations is measured by $\frac{1000}{5} = 200$ and that of the second set of observations by $\frac{1435}{41} = 35$. And we now see that as expected the first set has higher dispersion than the second.

The exact definition of the variance is as follows:

Definition: The variance of a given set of observations $x_1, x_2, \ldots, x_n$, of a quantitative variable is the number

$$\frac{(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + \ldots + (x_n - \bar{x})^2}{n}$$

where $\bar{x}$ denotes the arithmetic mean of the observations.

The symbol σ^2 is normally used for the variance. We shall sometimes write the variance of the observations $x_1, x_2, \ldots, x_n$ as $\sigma^2(x_1, x_2, \ldots, x_n)$ if we wish to draw attention to the fact that we are talking of the variance of the n observations $x_1, x_2, \ldots, x_n$.

The variance is a measure of dispersion about the mean value $\bar{x}$; large values of the variance indicating greater dispersion than small values. However, the variance, being based on the squares of the differences $x_i - \bar{x}$ will not be measured in the same units as x_i and $\bar{x}$. If, for example, the observations x_i, and hence the mean $\bar{x}$, are lengths measured in metres, the variance will be measured in square metres, a unit of area and not of length. For this reason, the dispersion about the mean is usually expressed in the form of the square root of the variance which is called the *standard deviation* of the set of observations. The standard deviation is usually denoted by the symbol σ (to show its relationship with the variance σ^2), or, in short by s.d.

The exact definition of the standard deviation is as follows:

Definition

The *standard deviation* of a given set of observations $x_1, x_2, \ldots, x_n$ is the number given by

$$\sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2}$$

where $\bar{x}$ is the arithmetic mean of the observations.

If we have a frequency table with m classes with each class defined by a *single* value x_i with frequency f_i, the variance will be defined by

$$\sigma^2 = \frac{\sum_{i=1}^{m} f_i(x_i - \bar{x})^2}{\sum_{i=1}^{m} f_i} \quad \text{where} \quad \bar{x} = \frac{\sum_{i=1}^{m} f_i x_i}{\sum_{i=1}^{m} f_i}.$$

If each class of the frequency table consists not of a single value but a number of values, we follow the approximation procedure which was followed for the calculation of the mean. Each class is replaced by a single value equal to the mid-value for that class and then the above procedure for single value classes is followed to get an *approximate* value of the variance.

Calculation of Variance: For the actual calculation of the variance, we do not use the formulae given in the definition. Instead we make use of the following *identity*:

$$\begin{aligned}
\sum_{i=1}^{n}(x_i-\bar{x})^2 &= \sum_{i=1}^{n}(x_i^2-2x_i\bar{x}+\bar{x}^2)\\
&= \sum_{i=1}^{n}x_i^2-2\bar{x}\sum_{i-1}^{n}x_i+n\bar{x}^2\\
&= \sum_{i=1}^{n}x_i^2-2\bar{x}n\bar{x}+n\bar{x}^2\\
&= \sum_{i=1}^{n}x_i^2-n\bar{x}^2=\sum_{i=1}^{n}x_i^2-\frac{(\sum x_i)^2}{n}
\end{aligned}$$

This has the advantage of reducing the number of arithmetic operations to be carried out (for example, we have one subtraction only in place of n subtractions), and of decreasing the chance of committing errors of calculation. It also reduces rounding-off errors.

So, for calculating the variance we shall use the formula

$$\sigma^2(x_1,x_2,\ldots,x_n)=\frac{1}{n}\left[\sum_{i=1}^{n}x_i{}^2-\frac{\left(\sum\limits_{i=1}^{n}x_i\right)^2}{n}\right]$$

In case of frequency tables with m classes, each class being defined by a single value x_i with frequency f_i, the variance will be calculated by the formula

$$\sigma^2=\frac{\sum\limits_{i=1}^{m}f_ix_i{}^2-\dfrac{(\sum\limits_{i=1}^{m}f_ix_i)^2}{\sum\limits_{i=1}^{m}f_i}}{\sum\limits_{i=1}^{m}f_i}$$

The same formula will be used to get the approximate value of the variance if the classes of the frequency table consist not of single values but of a number of values. We shall use the mid-value x_i of the class with frequency f_i in the above formula.

Example 17.1

The scores of batsman A were 38, 70, 48, 34, 42, 55, 63, 46, 54, 44.
The total being 494, we have $\bar{x}=49.4$. The differences $x_i-\bar{x}$ are

$$-11.4, 20.6, -1.4, -15.4, -7.4, 5.6, 13.6, -3.4, 4.6, -5.4.$$

Hence,

$$\sum(x_i - \bar{x})^2 = (11.4)^2 + (20.6)^2 + \ldots + (4.6)^2 + (5.4)^2$$

$$= 1126.64$$

and

$$\sigma^2 = \frac{1126.4}{10} = 112.64$$

$$\text{s.d.} = \sqrt{112.64} = 10.61$$

We have calculated the variance by using the definition of the variance. We now show how to obtain it by the formula for calculating it. In that formula, we have first of all to obtain the sums $\sum x_i$ and $\sum x_i^2$ (this is done in the table below), and then these sums are used to calculate the variance.

	x_i	x_i^2
	38	1444
	70	4900
	48	2304
	34	1156
	42	1764
	55	3025
	63	3969
	46	2116
	54	2916
	44	1936
Total	494	25530

$$\begin{aligned}
\frac{(\sum x_i)^2}{n} &= \frac{(494)^2}{10} \\
&= 24403.6 \\
\sum x_i^2 - \frac{(\sum x_i)^2}{n} &= 25530 - 24403.6 = 1126.4 \\
\sigma^2 &= \frac{1126.4}{10} = 112.64 \\
\text{Hence, s.d.} &= 10.61
\end{aligned}$$

Example 17.2

The scores (x_i) obtained by 21 children in an intelligence test are given along with their frequencies (f_i) in the first two columns of the table below. The next two columns are used to obtain the values of $f_i x_i$ and $f_i x_i^2$. The totals of these columns are then used to calculate the variance as shown. Note that the values $f_i x_i$ in the third column are obtained by multiplying the values x_i and f_i in the first two columns. The value of $f_i x_i^2$ is then obtained by multiplying the value of $f_i x_i$ in the third column by the value x_i in the first column. In this way, the sqauring of the values x_i is avoided and calculation time saved.

x_i	f_i	$f_i x_i$	$f_i x_i^2$
91	3	$91 \times 3 = 273$	$273 \times 91 = 24843$
92	2	184	16928
96	3	288	27648
97	2	194	18818
101	5	505	51005
103	3	309	31827
108	3	324	34992
Total	21	2077	206061

$$\frac{(\sum f_i x_i)^2}{\sum f_i} = \frac{(2077)^2}{21}$$

$$= \frac{4313929}{21} = 205425.19$$

$$\sum f_i x_i^2 - \frac{(\sum f_i x_i)^2}{(\sum f_i)} = 206061 - 205425.19 = 635.81$$

$$\sigma^2 = \frac{635.81}{21} = 30.28$$

$$\text{Hence, s.d.} = \sigma = 5.50$$

Example 17.3

The measurements (in mm) of the diameters of the heads of 107 screws gave the frequency table formed by the first two columns below. The third column gives the mid-value of the class from which the remaining columns as well as the value of the variance are obtained as in Example 17.2.

Diameter	*Frequency*	*Mid-value*	$f_i x_i$	$f_i x^2$
33-35	17	34	578	19652
36-38	19	37	703	26011
39-41	23	40	920	36800
42-44	21	43	903	38829
45-47	27	46	1242	57132
Total	107		4346	178424

$$\frac{(\sum f_i x_i)^2}{\sum f_i} = \frac{(4346)^2}{107} = 176520.71$$

$$\sum f_i x_i^2 - \frac{(\sum f_i x_i)^2}{\sum f_i} = 178424 - 176520.71$$

$$= 1903.29$$

$$\sigma^2 = \frac{1903.29}{107} = 17.79$$

Hence, s.d. $= \sigma = 4.22$.

17.12 Short-cut Method for $\bar{x}$ and σ^2

If the values of x_i or the mid-values x_i in the different classes of a frequency table are large, the calculations of $\bar{x}$ and σ^2 become quite lengthy and time-consuming. In the case of frequency table with equally spaced values or mid-values x_i, calculations can be simplified to a great extent by a simple method described as a 'Short-cut Method'. According to this method we first subtract a constant A from each x_i; A is generally chosen to be the middle x_i or a value near to the middle x_i in the frequency table. Each deviation $x_i - A$ is then divided by a suitable constant h which is generally the class interval in the frequency table.

Taking

$$u_i = \frac{x_i - A}{h} \quad \text{or} \quad x_i = A + hu_i,$$

we have

$$\bar{x} = \frac{\sum f_i x_i}{\sum f_i} = \frac{\sum f_i(A + hu_i)}{\sum f_i}$$

$$\therefore \quad \bar{x} = A + h\frac{\sum f_i u_i}{\sum f_i}$$

$$\text{and} \quad \sigma^2 = \frac{\sum f_i(x_i - \bar{x})^2}{\sum f_i}$$

$$= \frac{\sum f_i(A + hu_i - A - h\bar{u})^2}{\sum f_i}$$

$$\therefore \quad \sigma^2 = \frac{h^2\left(\sum f_i u_i^2 - \frac{(\sum f_i u_i)^2}{\sum f_i}\right)}{\sum f_i}$$

Let us work out below the Example 17.3 by using the above formula for $\bar{x}$ and σ^2:

Diameter	Frequency f_i	Mid-value x_i	$u_i = \frac{x_i - 40}{3}$	$f_i u_i$	$f_i u_i^2$
33-35	17	34	-2	-34	68
36-38	19	37	-1	-19	19
39-41	23	40	0	0	0
42-44	21	43	$+1$	$+21$	21
45-47	27	46	$+2$	$+54$	108
Total	107			$+22$	216

Here $A = 40$ and $h = 3$

$$\bar{u} = \frac{\sum f_i u_i}{\sum f_i} = \frac{22}{107} = 0.21$$

$$\bar{x} = 40 + 3 \times 0.21 = 40.63$$

$$\frac{(\sum f_i u_i)^2}{\sum f_i} = \frac{(22)^2}{107} = 4.52$$

$$\sum f_i u_i - \frac{(\sum f_i u_i)^2}{\sum f_i} = 216 - 4.52 = 211.48$$

$$\therefore \quad \sigma^2 = \frac{9 \times 211.48}{107} = 17.79$$

Hence, s.d. $= \sigma = 4.22$.

EXERCISE 17.3

1. Prove the identity

$$\sum_{i=1}^{n}(x_i - \bar{x})^2 = \sum_{i=1}^{n} x_i^2 - n\bar{x}^2 = \sum_{i=1}^{n} x_i^2 - \frac{(\sum_{i=1}^{n} x_i)^2}{n}$$

2. (a) What would be the variance of the scores of batsman A (in Section 17.11) if he had scored 10 runs more in each innings ?

(b) What would be the variance of the scores of batsman A (in Section 17.11) if he had scored 10 runs less in each innings ?

3. Suppose the observations $x_1, x_2, \ldots, x_n$ are changed to $x_1+a, x_2+a, \ldots, x_n+a$, where a is a given positive or negative number. Show that the variance remains unchanged. [Hint: The mean $\bar{x}$ becomes $\bar{x}+a$]

4. (a) The scores of 10 students in a test, in which the maximum marks were 50, were as follows:
   28, 36, 34, 28, 48, 22, 35, 27, 19, 41. Find the variance.
   (b) Later on the maximum marks were increased to 100, and accordingly each student's score was doubled. Find the variance of the new scores.

5. Given that σ^2 is the variance of the observations $x_1, x_2, \ldots, x_n$, prove that the variance of $ax_1, ax_2, \ldots, ax_n$, where a is *any* number different from zero, is $a^2\sigma^2$.
   [Hint: $\bar{x}$ becomes $a\bar{x}$ and $x_i - \bar{x}$ becomes $a(x_i - \bar{x})$]

6. The scores of 48 children in an intelligence test are shown in the frequency table below:

Score	*Frequency*
71	4
76	3
79	4
83	5
86	6
89	5
92	4
97	4
101	3
103	3
107	3
110	2
114	2

   Calculate the variance σ^2 and find out the percentage of children whose scores lie between $\bar{x}-\sigma$ and $\bar{x}+\sigma$.

(b) *Mean Deviation about the Median*

Let us now consider the problem of measuring the dispersion about the median. Once again we begin with the individual differences $x_i - M$ between the observations x_i and the median M. If the observations are close to the median those differences will be small in magnitude. If on the other hand the observations are far from median they will be large. Since M is the median of the observations x_i, half of the differences $x_i - M$ will be positive and half negative. Because of it, we cannot just take the average of these values as a measure of dispersion. We had a similar problem in the case of the variance where we squared each difference $x_i - \bar{x}$ to make all of them positive. Now we follow a slightly different method. We simply ignore the negative sign if $x_i - M$ is negative. In other

words, we replace each difference $x_i - M$ by its *absolute value* $|x_i - M|$. Thus if $x_i - M$ is 3.5, we have $|x_i - M| = 3.5$ and if $x_i - M$ is equal to -4.01, we take $|x_i - M| = 4.01$. We add these absolute values to get the sum

$$|x_1 - M| + |x_2 - M| + \ldots + |x_n - M| = \sum_{i=1}^{n} |x_i - M|$$

which is an indicator of the amount of dispersion of the observed values about the median. This sum is then divided by the number of observations n to get the mean deviation about the median. The division by n is done for the same reason for which $\sum_{i=1}^{n}(x_i - \bar{x})^2$ was divided by n to get the variance measure of dispersion about the mean.

The definition of the *mean deviation about the median* (or just *mean deviation* in short) is as follows:

The mean deviation of a given set of observations $x_1, x_2, \ldots, x_n$, about their median M is the number

$$\frac{|x_1 - M| + |x_2 - M| + \ldots + |x_n - M|}{n}$$

The individual difference in absolute value, $|x_i - M|$ are called *deviations* of the observations from M and the mean deviation, as the name suggests is their average. Since, unlike the variance, the mean deviation is measured in the same units as the observations, no further modifications are needed and the mean deviation can be used as it is as a measure of dispersion about the median.

We could also have measured the dispersion about M by obtaining the average

$$\frac{(x_1 - M)^2 + (x_2 - M)^2 + \ldots + (x_n - M)^2}{n}$$

and then taking its square root. Similarly, the dispersion about the mean $\bar{x}$ could have been measured by the average

$$\frac{|x_1 - \bar{x}| + |x_2 - \bar{x}| + \ldots + |x_n - \bar{x}|}{n}$$

However, we follow different methods while measuring dispersion about the mean $\bar{x}$ and about the median M. The reasons for doing so cannot be explained at this stage and will be learnt by you when you study the subject at a more advanced level. For the present it is enough to remember that the variance (or standard deviation) is used to measure the dispersion about the mean, and the mean deviation is used to measure the dispersion about the median.

Calculation of the mean Deviation: Suppose the observations are $x_1, x_2, \ldots, x_n$, and their median value is M. If we calculate the mean deviation according to its definition, we have to obtain the values of $x_i - M$ for all the observations. However, this is not necessary and we can do the calculations more easily and in a shorter time as follows:

(i) Separate the observations x_i which are $\geq M$ from those x_i which are $< M$.

(ii) Count the number (n_1) of the observations which are $\geq M$, and obtain their total (s_1).

(iii) Count the number (n_2) of the observations which are $< M$, and obtain their total (s_2).

(iv) Check that $n_1 + n_2 = n$.

(v) Obtain the value of the mean deviation about the median as

$$\frac{(s_1 - s_2) - (n_1 - n_2)M}{n_1 + n_2}$$

If the observations are given in the form of a frequency table with m classes in which each class consists of a single value x_i with corresponding frequency f_i, we proceed as follows:

(i) Take the observations x_i which are $\geq M$; obtain the sum (n_1) of their frequencies f_i and the sum (s_1) of the products $f_i x_i$ for them.

(ii) Take the observations x_i which are $< M$; obtain the sum (n_2) of their frequencies f_i and the sum (s_2) of the products $f_i x_i$ for them.

(iii) Check that $n_1 + n_2$ equals the total frequency.

(iv) Obtain the mean deviation as

$$\frac{(s_1 - s_2) - (n_1 - n_2)M}{n_1 + n_2}$$

Example 17.4

We consider the scores
38, 70, 48, 34, 42, 55, 63, 46, 54, 44 in ten innings of batsman A. If we rearrange the scores in increasing order, we get
34, 38, 42, 44, 46, 48, 54, 55, 63, 70. The fifth value is 46 and the sixth is 48. Hence we take their average 47 as the median M.

With this value of M, the mean deviation according to the definition is given by

$$\begin{aligned} &\frac{1}{10}\left[|34 - 47| + |38 - 47| + \ldots + |70 - 47|\right] \\ &= \frac{1}{10}\left[13 + 9 + 5 + 3 + 1 + 1 + 7 + 8 + 16 + 23\right] \\ &= \frac{1}{10}[86] = 8.6 \end{aligned}$$

If we use the alternative method of calculating the mean deviation, we find that

(i) Observations 48, 54, 55, 63, 70 are $\geq M$ so that $n_1 = 5$ and $s_1 = 290$

(ii) Observations 34, 38, 42, 44, 46 are $< M$ so that $n_2 = 5$ and $s_2 = 204$. Hence, mean deviation is given by

$$\frac{(290 - 204) - (5 - 5)47}{5 + 5} = \frac{86}{10} = 8.6$$

In this particular example, we have $n_1 = n_2$ so that the term $(n_1 - n_2)M$ was reduced to zero. However, we need not necessarily have $n_1 = n_2$ as will be seen in Example 17.5.

Example 17.5

Suppose the observations when arranged in increasing order are:
3, 3, 5, 9, 10, 12, 12, 12, 18, 21, 21, for which median $M = 12$. We now have

$$s_1 = 12 + 12 + 12 + 18 + 21 + 21 = 96$$

$$s_2 = 3 + 3 + 5 + 9 + 10 = 30$$

$$n_1 = 6$$

$$n_2 = 5$$

Thus, we have mean deviation

$$= \frac{(96 - 30) - (6 - 5)12}{6 + 5} = \frac{66 - 12}{11} = \frac{54}{11}$$

If we follow the definition, we get the mean deviation as

$$\frac{1}{11}[|3 - 12| + |3 - 12| + |5 - 12| + |9 - 12| + |10 - 12| + |12 - 12| + |12 - 12| + |12 - 12|$$
$$+|18 - 12| + |21 - 12| + |21 - 12|]$$
$$= \frac{1}{11}[9 + 9 + 7 + 3 + 2 + 0 + 0 + 0 + 6 + 9 + 9] = \frac{54}{11}$$

EXERCISE 17.4

. The lengths (in cm) of 30 small pieces of cloth left after a day's sale in a cloth shop were:
42.7, 98.5, 29.3, 29.1, 89.2, 69.2, 70.8, 5.7, 22.8, 49.1, 44.3, 79.5, 66.4, 14.2, 10.6, 41.1, 91.8, 86.4, 49.4, 85.5, 56.1, 49.6, 72.8, 59.4, 6.4, 32.9, 62.6, 37.3, 56.0, 33.1
Find the mean deviation d about the median M. See how many observations are between $M - 2d$ and $M + 2d$.

2. . The areas (in square metres) of the different fields in a village were as follows:
67.1, 74.5, 94.9, 98.2, 15.3, 91.5, 80.2, 42.3, 62.0, 75.3, 57.0, 88.0, 49.4, 34.1, 56.1, 31.2, 36.1, 83.2, 37.8, 80.3, 85.6, 53.9, 69.9, 71.9, 91.5, 98.3, 42.2, 32.6, 91.6, 40.8, 35.5.
Find the mean deviation d about the median M. See how many observations are between $M - d$ and $M + d$, and how many (in per cent) are between $M - 2d$ and $M + 2d$.

CHAPTER 18

Linear Programming

18.1 Linear Constraints

Suppose we have a sum of Rs 35 with us with which we wish to purchase pencils which cost Rs 2 each, and notebooks which cost Rs 3 each. How many pencils and notebooks can we buy? If we buy pencils only, we can buy at the most 17 pencils and will have Re 1 left; if we buy notebooks only, we can buy at the most 11 notebooks and will have Rs 2 left with us. We can also buy 5 pencils and 8 notebooks at a total cost of Rs $[(5 \times 2) + (8 \times 3)] =$ Rs 34 and will have Re 1 left, or, we can buy 7 pencils and 7 note-books, in which case we will have spent the total amount of Rs 35 which we had with us. Obviously, we cannot buy 8 pencils and 7 notebooks since their total cost is Rs 37 which is more than what we can spend. We thus see that we have many choices regarding the numbers of pencils and notebooks we can purchase without requiring more than Rs 35.

Let us denote the number of pencils by x, and that of notebooks by y. Obviously, x and y can take only positive and integral values. The problem we have been discussing can be put in mathematical language as follows:

Find positive integers x and y such that

$$2x + 3y \leq 35$$

The solutions to this problem, that is all possible pairs (x, y) which satisfy the above conditions can be graphically presented if we use our knowledge of coordinate geometry. We know that the equation $2x + 3y = 35$ represents a straight line. This line divides the plane into two half-planes. One of these half planes precisely contains points (x, y) for which $2x + 3y < 35$ and the other those for which $2x + 3y > 35$. The two regions A and B are shown in Fig. 18.1. In which of these two regions is $2x + 3y < 35$? We can try any point in one of the regions. Suppose we try the origin $(0, 0)$ which is in the region A. If we put $x = 0, y = 0$ in $2x + 3y$, we get 0 which is less than 35. So $(0, 0)$ satisfies

$$2x + 3y < 35$$

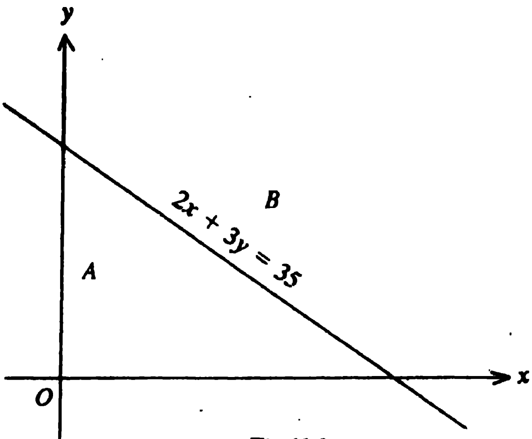


Fig 18.1

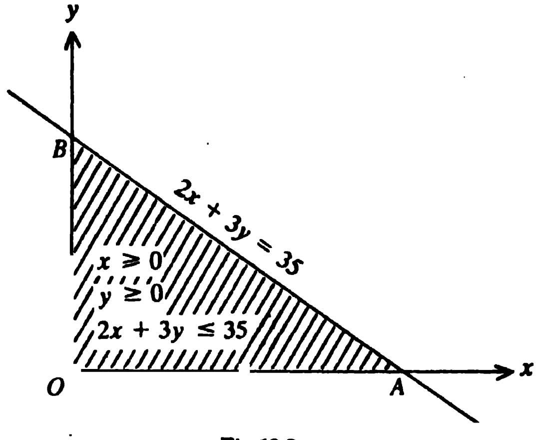


Fig 18.2

The region A satisfies $2x+3y < 35$ and B represents, $2x+3y > 35$. Since in our problem, we want x and y to be positive integers (x and y can be zero but cannot be negative), we see that the solutions to our problem are just those points (x, y) whose co-ordinates are integers and which fall in the shaded region OAB (including the boundaries) in Fig.18.2.

If we buy pencils only, the solution will be located on OA and if we buy notebooks only, the solution will lie on OB. If we are looking for a solution which completely utilises the money available (e.g., $x = 4, y = 9; x = 1, y = 11; x = 7, y = 7, x = 10, y = 5$, etc.), the solutions will be found on the line segment AB.

If we have to buy a minimum of 2 pencils and 3 notebooks, then our choice of x and y should be such that we also have $x \geq 2$ and $y \geq 3$, in addition to the requirement $2x + 3y \leq 35$. The points (x, y) for which y is non-negative and $x \geq 2$, and for which x is non-negative and $y \geq 3$ are shown in Figs 18.3 and 18.4.

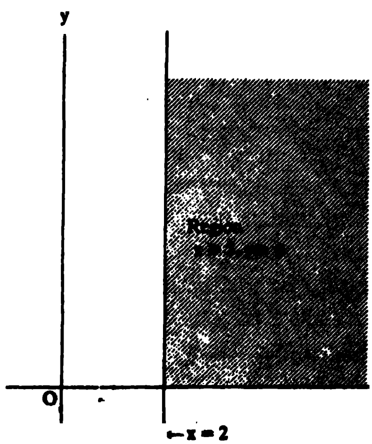


Fig 18.3

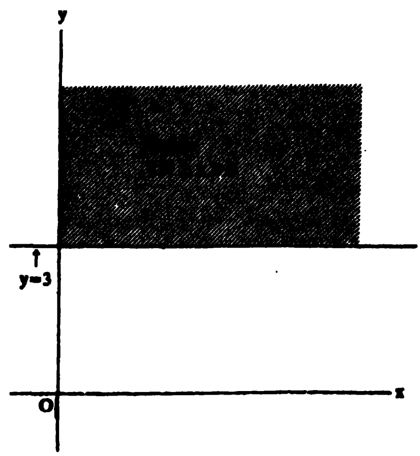


Fig 18.4

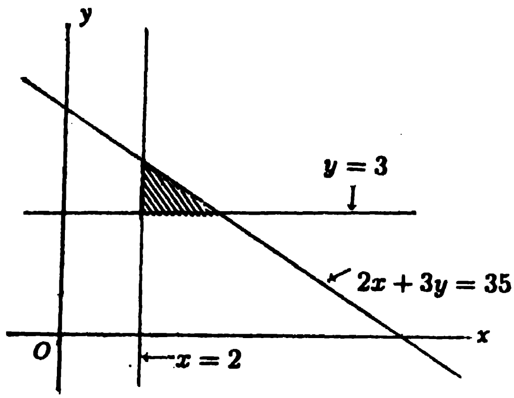


Fig 18.5

With this added restriction that we should have $x \geq 2$ and $y \geq 3$, the solution now are pairs of integers for which the point (x, y) lies in the shaded region shown in Fig.18.5, which is bounded by the lines $2x + 3y = 35, x = 2$, and $y = 3$.

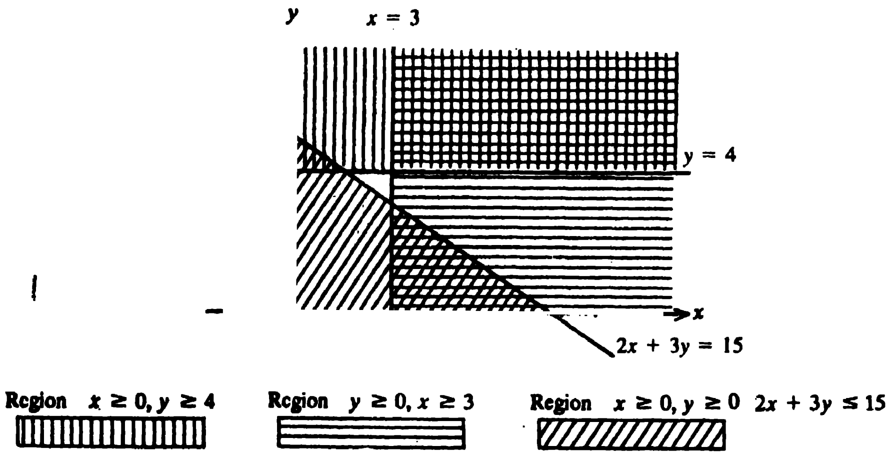


Fig 18.6

It can happen at times that the conditions we have laid down are such that they cannot be satisfied. In such cases there will be no solution to the problem. For example, suppose we wish to purchase at least 3 pencils and 4 notebooks and do not want to spend more than Rs 15. Clearly, this is impossible. The geometrical representation of these conditions is shown in Fig.18.6. We see that there is no region where all the conditions are satisfied. In such cases we say that there is no solution, or that the solution set is *empty*.

The conditions $2x + 3y \leq 15, x \geq 3$, etc. are also called *constraints* because they restrict our freedom of choice of the values x and y. The constraints we have used are

of the type called *linear constraints* because they are expressed by means of *linear functions*.

An important property of linear constraints on two variables x and y is that the set of points (x, y) for which the constraints are satisfied is either empty, or is a region bounded by straight lines (i.e. a polygon), or an unbounded region with straight line boundaries.

Example 18.1
The constraints

$$\begin{aligned} x + y &\le 5 \\ 4x + y &\ge 4 \\ x + 5y &\ge 5 \\ x &\le 4 \\ y &\le 3 \end{aligned}$$

have as solution set the shaded area (a polygon) in the Fig 18.7. This area is bounded by the five lines

$$x + y = 5, 4x + y = 4, x + 5y = 5, x = 4, \text{ and } y = 3$$

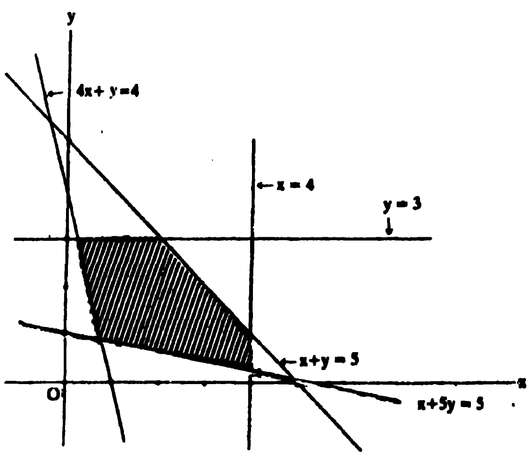


Fig 18.7

Example 18.2
The solution set of the constraints

$$\begin{aligned} 3x + 4y &\ge 12 \\ y &\ge 1 \\ x &\ge 0 \end{aligned}$$

is the shaded portion shown in Fig. 18.8. It is bounded on three sides by the lines $x = 0, 3x + 4y = 12$ and $y = 1$, but is unbounded as the constraints are satisfied by arbitrarily large positive values of x and y.

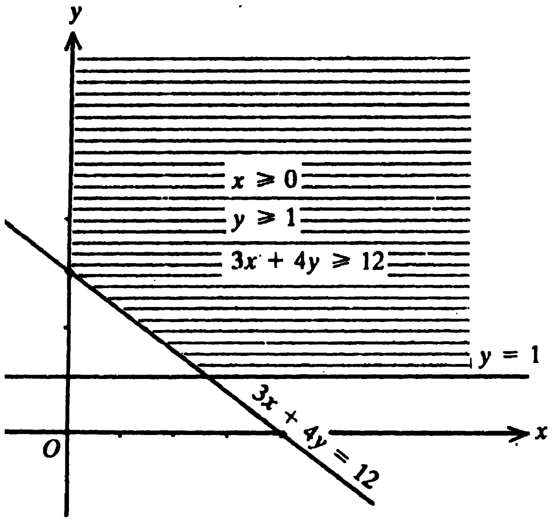


Fig. 18.8

Example 18.3

If in example 18.2 we add one more constraint $4x + 7y \leq 28$, the solution set (shaded area in Fig. 18.9) is no longer unbounded, but is a quadrilateral (i.e., a polygon) bounded by the lines $x = 0, 3x + 4y = 12, y = 1$ and $4x + 7y = 28$.

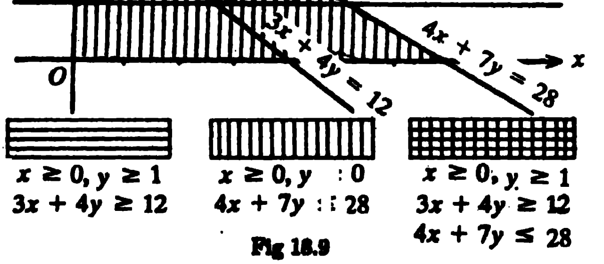


Fig 18.9

Example 18.4

This time if we add to example 18.2 the constraint $x + 2y \leq 3$, the solution set is empty as is clear from Fig. 18.10.

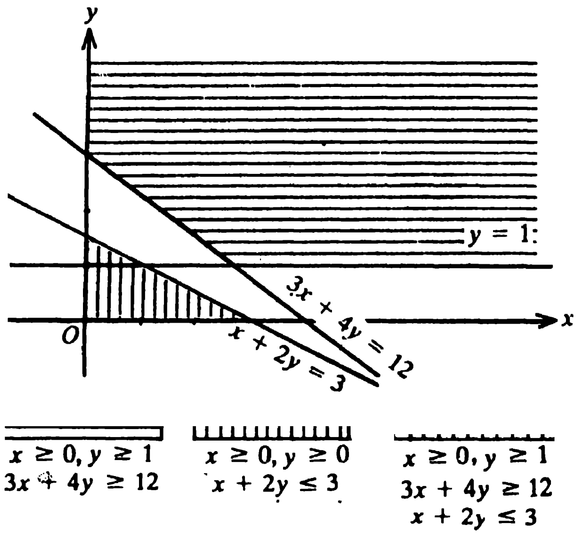


Fig. 18.10

EXERCISE 18.1

1. Draw the diagram of the solution set of the linear constraints

$$\begin{aligned} 2x + 3y &\leq 6 \\ x + 4y &\leq 4 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

2. Exhibit graphically the solution set of the linear constraints

$$\begin{aligned} x + y &\geq 1 \\ y &\leq 5 \\ x &\leq 6 \\ 7x + 9y &\leq 63 \\ x, y &\geq 0 \end{aligned}$$

3. Verify that the solution set of the following linear constraints is empty:

$$\begin{aligned} x - 2y &\geq 0 \\ 2x - y &\leq -2 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

4. Verify that the solution set of the first two constraints of problem above is not empty, and that it is unbounded.

5. Find the linear constraints for which the shaded area in Fig. 18.11 is the solution set.

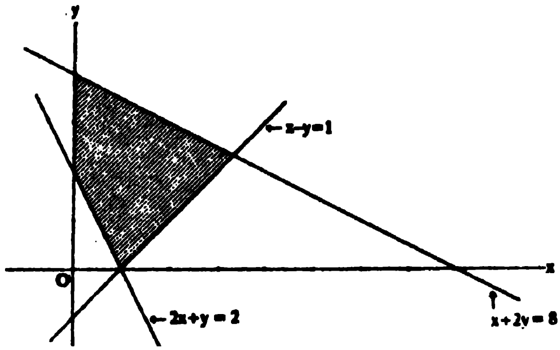


Fig. 18.11

6. Find the linear constraints for which the shaded area in Fig. 18.12 is the solution set.

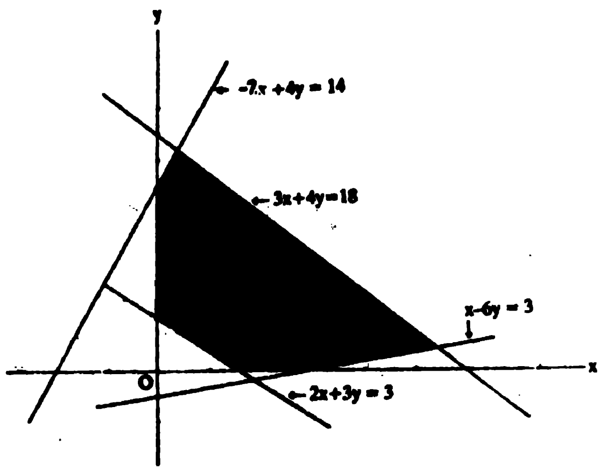


Fig 18.12

18.2 Linear Programming

We now look at the problem of purchase of notebooks and pencils from the point of view of a shopkeeper who buys goods in order to sell them at a profit. The number of pencils and notebooks he can purchase will depend on the amount of money he can invest. He has also to make sure that he does not buy more than what he can sell. Finally, he will try to see that the number of pencils and notebooks which he purchases is such as to give him the maximum amount of profit on sale.

Suppose the pencils cost him Re 0.75 each, and the notebooks Rs 3 each. If he has Rs 100 to invest, and x and y denote, respectively, the numbers of pencils and notebooks purchased, then we have

$$0.75x + 3y \leq 100$$

Now suppose he is able to sell all the pencils and notebooks purchased by him, the pencils at Re 1 each, and the notebooks at Rs 3.50 each. Thus he makes a profit of Rs 0.25 on each pencil and a profit of Rs 0.50 on each notebook. Since he makes a larger profit on notebooks, we could be tempted to suggest that the shopkeeper should only purchase notebooks. However, that would not be a correct advice. For, by investing Rs 3 in the purchase of 4 pencils he would make a total profit of Re 1, whereas by investing it in the purchase of 1 notebook his profit would only be Re 0.50. Some possible values of x and y with the resulting profit are given in the table below:

x	y	*Cost (Rs)*	*Profit (Rs)*
1	33	99.75	16.75
5	32	99.75	17.25
53	20	99.75	23.25
21	24	87.75	17.25
33	20	84.75	18.25
133	–	99.75	33.25

We see that his profit is maximum if he invests all his capital in the purchase of pencils only. Then he utilises almost all his capital and makes a profit of Rs 33.25. Suppose the shopkeeper knows from past experience that the sales of notebooks cannot exceed 25, and therefore he should not purchase more than 25 notebooks. Suppose he also knows that customers of notebooks usually also buy pencils, and that if pencils run out of stock it becomes difficult to sell notebooks. Accordingly, he decides that to begin with the number of pencils in stock should be at least four times the number of notebooks. Thus, his choice of x and y is now determined by the following conditions:

$$\begin{aligned} 0.75x + 3y &\leq 100 \\ y &\leq 25 \\ x &\geq 4y \end{aligned}$$

Of course, x and y should be positive integers.

The possible values of x and y which satisfy these conditions will be such that the point (x, y) lies in the shaded region in Fig. 18.13

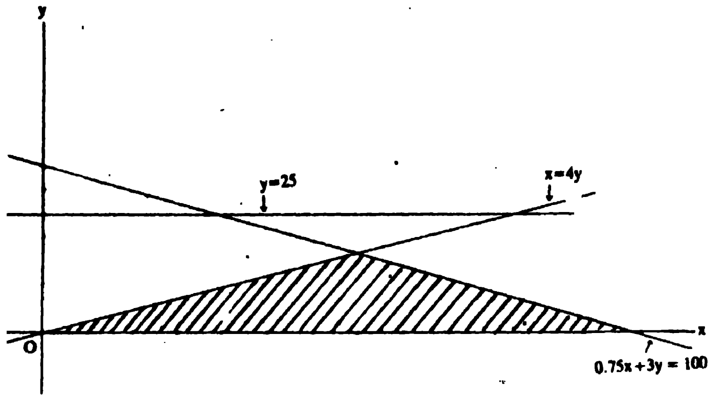


Fig. 18.13

Out of all the points (x, y) in the shaded region, the shopkeeper has to choose that point, or those points, for which the profit

$$0.25x + 0.5y$$

has the maximum value. Some possible values of x and y with the resulting profit are given in the table below:

x	y	*cost (Rs)*	*Profit (Rs)*
80	13	99	26.5
84	11	96	26.5
116	4	99	31
129	1	99.75	32.75

We notice that the shopkeeper gets the same profit (Rs 26.5) if he invests Rs 99 or Rs 96. He would no doubt prefer to invest Rs 96 instead of Rs 99 if he knows that in both cases his profit is the same. In view of this the shopkeeper can look at the problem in a different manner. He can fix a minimum amount of profit to be made and then try to achieve it with as small an investment as possible.

Suppose the shopkeeper decides that his profit should not be below Rs 30. He has now to choose x and y in such a manner that he can make a profit of Rs 30 or more keeping his investment as low as possible. Now, the variables x and y have to satisfy only one condition, namely

$$0.25x + 0.5y \geq 30$$

in addition to being non-negative integers. Out of all such pairs of values (x, y), he chooses that for which

$$0.75x + 3y$$

has the smallest value.

The two situations described above illustrate the type of problems called linear programming problems. The term *linear programming* refers to the special methods evolved for solving such problems. We give below a brief description of these methods.

Mathematical Model of Linear Programming Problems

We have two variables x_1 and x_2 which are non-negative (i.e., $x_1 \geq 0, x_2 \geq 0$). These variables have also to satisfy a number of *linear* constraints. Since a linear constraint $3x_1 + 4x_2 \geq 5$ can also be expressed as $-3x_1 - 4x_2 \leq -5$, we write all the constraints of the problem in the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &\leq b_1 \\ a_{21}x_1 + a_{22}x_2 &\leq b_2 \end{aligned}$$

and so on. We are also given a *linear* function

$$c_1x_1 + c_2x_2$$

whose value is to be maximised (or minimised) subject to the given constraints. In other words, out of all pairs (x_1, x_2) satisfying the constraints we have to choose those which give a maximum (or minimum) value to the given linear function. A linear programming problem when expressed in mathematical form looks as follows:
Find $\max(3x_1 - 2x_2)$ given that

$$\begin{aligned} x_1 + x_2 &\leq 1 \\ 3x_1 - x_2 &\leq -3 \\ x_1, x_2 &\geq 0 \end{aligned}$$

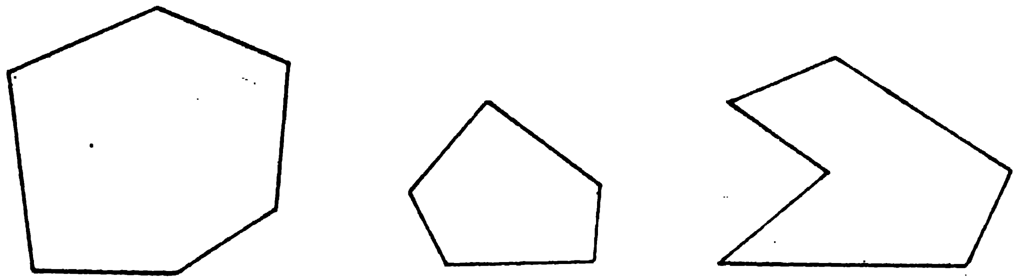
or it may be:
Find $\min(1.5x_1 + 2.5x_2)$ given that

$$\begin{aligned} -x_1 - 3x_2 &\leq -3 \\ x_2 &\leq 5 \\ x_1, x_2 &\geq 0 \end{aligned}$$

The set of pairs of values (x_1, x_2) which satisfy the linear constraints of the problem is called the set of *feasible solutions* to the problem. The linear function whose value is to be maximised (or minimised) is called the *objective function*.

18.3 Solution of a Linear Programming Problem

In most linear programming problems the set of feasible solutions is a *polygon* in the positive(first) quadrant, that is, a closed figure bounded by straight lines. The set is also a *convex set* which only means that if you take any two points in the set the line joining them also lies in the set. The set of feasible solutions is thus a *convex ploygon*. In Fig. 18.14 polygons at (i) and (ii) are convex, while that at (iii) is not.



(i) (ii) (iii)

Fig. 18.14

Once we have determined this convex polygon, we have to select a point, or points in it which will make the value of the objective function maximum (or minimum). Since the objective function is linear, we do not have toconsider all the points inside the polygon to locate the point which maximises (or minimises) the objective function. It is known from general mathematical theory that it is enough to look at the values of the objective function at the vertices of the set of feasible solutions. The largest of these values is the maximum value of the objective function, and the smallest of these values is the minimum. It may happen that two vertices give the same value of the objective function which also happens to be the largest value. In that case all the points on the line joining these two vertices will give the maximum value of the objective function. Similar is the case when two vertices will give the same value which is the smallest of all the values. Then all the points on the line joining these two vertices will give the minimum value to the objective function.

The different methods of solving linear programming problems make use of the above property of linear functions. One begins by obtaining the value of the objective function at one of the vertices and then moving to another vertex which increases the value of the objective function. The procedure is continued till one reaches the maximum value. The procedure for obtaining the minimum value is similar.

We now describe a geometrical method to solve a linear programming problem. This method locates the vertex of maximum (or minimum) value without having to evaluate the value of the objective function at any other vertex. Consider the following linear programming problem. Maximise $x + 2y$ under the constraints

$$\begin{aligned} 2x + 3y &\leq 6 \\ x + 4y &\leq 4 \\ x, y &\geq 0 \end{aligned}$$

The set of feasible solutions is represented by the shaded area $OPQR$ in Fig. 18.15. (See Exercise 18.1 problem No.1)

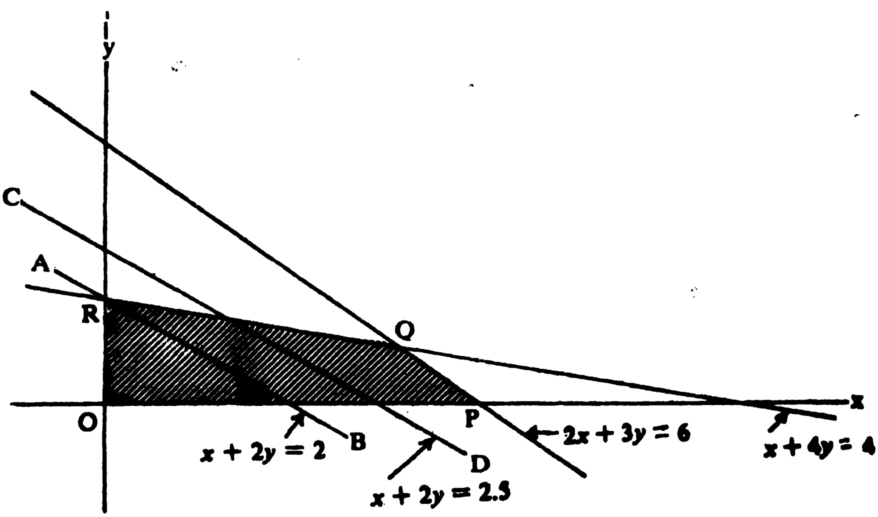


Fig. 18.15

$x = 1, y = 0.5$ is a feasible solution and for this solution the objective function has value 2. The solution (0.8,0.6) also gives the value 2 to the objective function. In fact all the solutions falling on the line $x + 2y = 2$ (the line AB in the figure) will give the same value 2 to the objective function. Similarly, the objective function has value 2.5 for all solutions (x, y) for which $x + 2y = 2.5$. These solutions fall on the line CD which is parallel to AB and away from the origin. Thus, as we move the line AB parallel to itself and away from the origin, we get feasible solutions giving larger and larger values to the objective function. The value of the objective function keeps increasing till we get a line parallel to AB and passing through the vertex Q of the polygon of feasible solutions. Any further displacement of the line takes it out of the polygon $OPQR$ so that the value

of the objective function corresponding to the line is no longer attained by any feasible solution. The maximum value of the ojective function takes place at the vertex $Q(\frac{12}{5}, \frac{2}{5})$ and equals $\frac{12}{5} + 2 \times \frac{2}{5} = \frac{16}{5} = 3.2$. If we move the line AB parallel to itself but towards the origin, the value of the objective function keeps decreasing. The minimum value is reached at the vertex O and equals zero.

As another example consider the set of feasible solutions of the constraints.

$$\begin{aligned} -x - y &\leq -1 \\ 7x + 9y &\leq 63 \\ x &\leq 6 \\ y &\leq 5 \\ x, y &\geq 0 \end{aligned}$$

It is given by the polygon $ABCDEF$ in Fig. 18.16. (See Exercise 18.1, problem No.2).

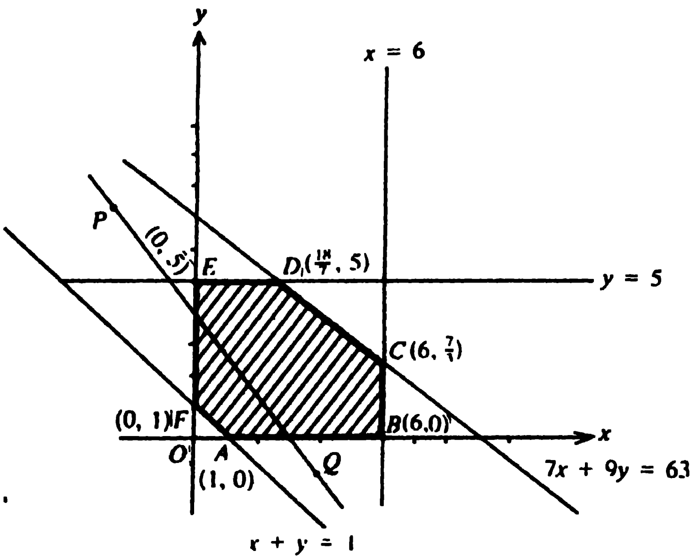


Fig. 18.16

To find the maximum or minimum value of the objective function $4x + 3y$ we move the line PQ (with slope $-\frac{4}{3}$) parallel to itself. When it is moved away from the origin, the line continues to cross the polygon $ABCDEF$ till it reaches the vertex C. The maximum value of the objective function is, therefore, given by the solution represented by the vertex C. Similarly, by moving the line towards the origin, we see that the

minimum value of the objective function is determined by the vertex F. The coordinates of the six vertices of the polygon $ABCDEF$ are given by

$$A:(1,0),\ B:(6,0),\ C:(6,\frac{7}{3})$$
$$D:(\frac{18}{7},5),\ E:(0,5),\ F:(0,1)$$

The values of the objective function $4x+3y$ at these vertices are as follows:

$$A:4,\ B:24,\ C:31,$$
$$D:25\frac{2}{7},\ E:15,\ F:3$$

and we verify that among the vertices, the maximum value of the objective function is at C and minimum is at F.

EXERCISE 18.2

1. Find the maximum and minimum values of $5x+2y$ subject to the constraints

$$\begin{aligned} -2x-3y &\le -6 \\ x-2y &\le 2 \\ 6x+4y &\le 24 \\ -3x+2y &\le 3 \\ x,y &\ge 0 \end{aligned}$$

2. Find the maximum and minimum values of $2x+y$ subject to the constraints

$$\begin{aligned} x+3y &\ge 6 \\ x-3y &\le 3 \\ 3x+4y &\le 24 \\ -3x+2y &\le 6 \\ 5x+y &\ge 5 \\ x,y &\ge 0 \end{aligned}$$

3. Find the minimum value of $3x+5y$ subject to the constraints

$$\begin{aligned} -2x+y &\le 4 \\ x+y &\ge 3 \\ x-2y &\le 2 \\ x,y &\ge 0 \end{aligned}$$

4. If a young man rides his motor-cycle at 25 km per hour, he has to spend Rs 2 per km on petrol; if he rides it at a faster speed of 40 km per hour, the petrol cost increases to Rs 5 per km. He has Rs 100 to spend on petrol and wishes to find what is the maximum distance he can travel within one hour. Express this as a linear programming problem and then solve it.

5. Every gram of wheat provides 0.1 g of proteins and 0.25 g of carbohydrates. The corresponding values for rice are 0.05 g and 0.5 g respectively. Wheat costs Rs 2 per kg and rice Rs 8. The minimum daily requirements of proteins and carbohydrates for an average child are 50 g and 200 g respectively. In what quantities should wheat and rice be mixed in the daily diet to provide the minimum daily requirements of proteins and carbohydrates at minimum cost. (The protein and carbohydrate values given here are fictitious and may be quite different from the actual values.)

CHAPTER 19

Algorithms and Flowcharts

19.1 Computers

The advent of computer is affecting practically every activity of human life. Whether it is reservation in railways, managing a super market, or clearing cheques in banks, we have begun using computer as a tool everywhere. Now-a-days in India, every big organisation is using computer for its day-to-day working. We use computer to locate underground natural gas and oil, to launch a satellite, and so on. In fact, the invention of computer is one of the finest contributions of science to society.

The development of the ideas leading to the present-day computer dates back to the nineteenth century. Some mathematicians like Charles Babbage (1791-1871), Alan Turing (1912-1954) and John Von Neumann (1903-1957) contributed significantly to the concept of a computing machine. Mathematics and computer enrich each other. The present-day computer has become a powerful tool to solve some mathematical problems that remained unsolved hitherto.

A computer, however, is not a machine that can do more mathematical operations than what a human being can do. But it can do some operations much faster and can handle large amount of data. You may be surprised to know that the present-day computer can perform a million additions per second. It can perform all operations such as addition, subtraction, multiplication, division and comparison.

There has been a variety of computers, depending on their purpose and capabilities. One common feature is that every computer is constituted of three major components: (1) Input and Output component, (2) Central Processing Unit, and (3) Memory.

Depending on the nature of these three components, the computers vary. All the computers do the same functions with varying degrees of speed and limits.

However, when a problem is given to the computer, it cannot automatically decide the operations and the operands. It is necessary for a person, desirous to use a computer for solving a problem, to instruct the computer properly to choose the operations and the operands. This requires a mode of communication with a computer. Imagine that a Chinese wants to tell a story to an Englishman. Though the Chinese knows the story,

unless he knows a language that the Englishman understands, he cannot actually communicate with him. Similarly, if any instruction is to be given to the computer, it is necessary that instructions are delivered in a language that the computer understands. Such languages are called Programming Languages: FORTRAN, BASIC, PASCAL, COBOL, etc. Irrespective of language, what is equally important, is to prepare the sequence of instructions to solve the specific problem. That is what we call an *algorithm*.

19.2 Algorithms

An algorithm is nothing but a sequence of simple steps to solve the problem at hand. It is a step-by-step procedure leading to the solution.

For example, to solve the equation $ax + b = c$, we may write:

Step 1: Subtract b from c
Step 2: Divide this answer by a, if $a \neq 0$
Step 3: This answer represents the value of x as the solution

Some characteristic features of an algorithm are:

(i) It is precise. There can be no ambiguity in a computer algorithm. Each step of the execution must be uniquely defined and may depend only on the inputs, the previous steps and the internal capabilities of the machine.

(ii) It is sufficiently detailed and precise to allow execution by the processor. For example, "find the area of the triangle", "solve the quadratic equation", etc. are instructions that require more details. In an algorithm, we have to incorporate how the area is to be found, or how the roots of the quadratic equation are to be determined. There is no room left for the creative imagination of the executor. But while planning an algorithm, some details may be suppressed.

(iii) It is in a definite order. The instructions in an algorithm are to be given in the order in which they are to be performed. The machine carries out these instructions, one after the other.

We say an algorithm, which is designed for a specific problem, is correct if it serves for all instances of the problem. Designing and studying computer algorithm occupies a unique and central position in the study of computer. Designing of algorithms is nothing but planning. Though algorithms occur in some of the earliest of human records, it is the rise of computers that has led to the widespread study of algorithm for their own sake.

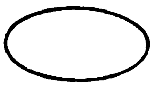
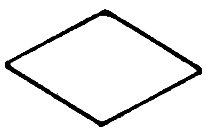
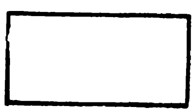
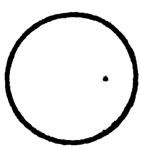
19.3 Flowcharts

Before beginning to solve a problem, it is frequently useful to do some preliminary planning. A graphic representation of such a plan is called a *flowchart*. The flowcharts constitute schematic pictures that we intend to implement in the program. The flowchart

helps to break a big job down into many small pieces of the job and to represent them pictorially, showing the order of instructions.

A flowchart consists of some boxes, linked by arrows. In each box, certain action to be carried out, is mentioned. Arrows on the lines connecting the boxes indicate the direction in which we should proceed.

The boxes are of different shapes as shown below

Shape of the box	*Meaning*
 a stretched ellipse	Terminal box: To begin or end a program.
 or  a parallelogram	Input/Output box: The data fed into the computer and the print out given by the computer
 a rhombus or a diamond	Decision box: Computer is to decide among alternative instructions
 a rectangle	Assignment/Calculation box: Computer is to assign some values for the variables, or is to perform some operations like addition, etc.
 a circle	Connector box: To come from or to go to another part of the chart

We start with an easy example.

Example 19.1
Problem: Find the modulus of a complex number given in the form (X, Y).
Method: The modulus of (X, Y) is given by the formula $r = \sqrt{X^2 + Y^2}$.

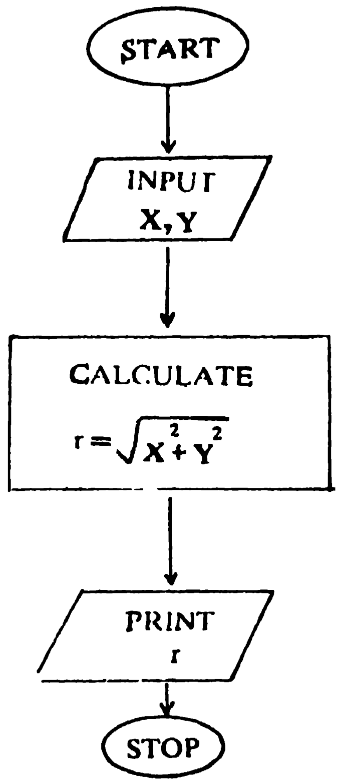


Fig. 19.1

Example 19.2
Problem: Given two complex numbers $Z_1 = (X_1, Y_1)$ and $Z_2 = (X_2, Y_2)$; compute $\frac{Z_1}{Z_2}$.

Method: If $\frac{Z_1}{Z_2}$ is (X_3, Y_3), then

$$X_3 = \frac{X_1X_2 + Y_1Y_2}{X_2^2 + Y_2^2}$$

and
$$Y_3 = \frac{Y_1X_2 - Y_2X_1}{X_2^2 + Y_2^2}$$

When $Z_2 = 0$, that is when both X_2 and Y_2 are zeroes, we do not calculate $\frac{Z_1}{Z_2}$.

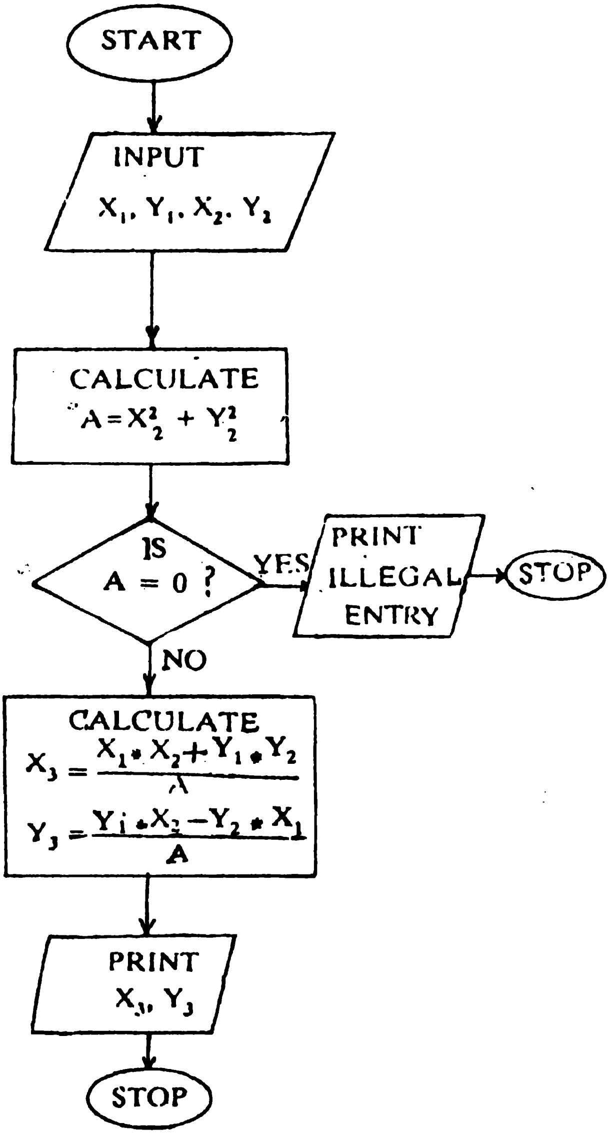


Fig 19.2

19.4 Loops

In this Section you will be taught how to assign different values for a variable, and create a loop while writing an algorithm. This is best explained by means of an example.

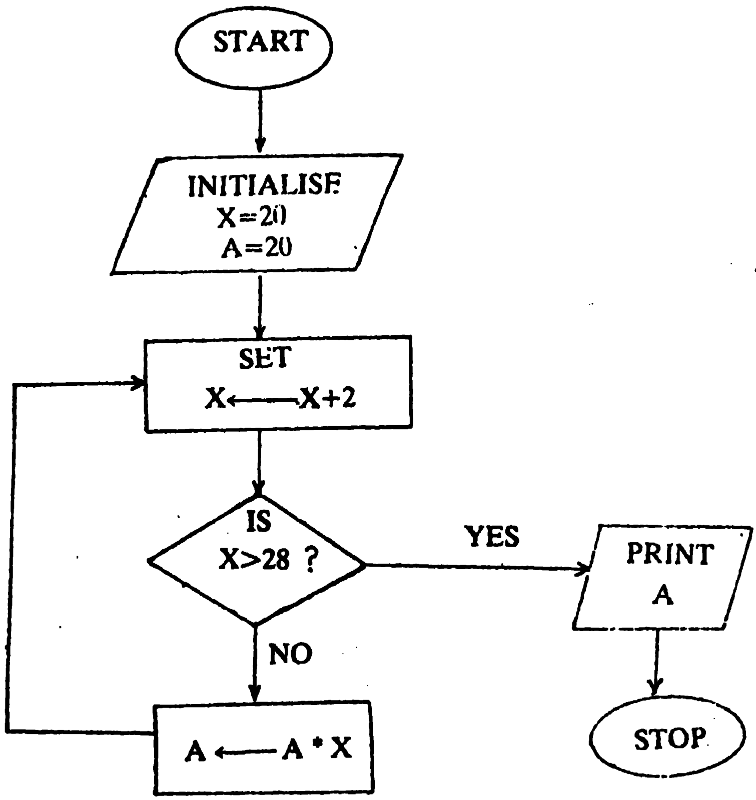


Fig. 19.3

Example 19.3

Problem: Multiply 20 × 22 × 24 × 26 × 28

Method: First multiply 20 × 22.
Then multiply this answer by 24.
Then multiply this answer by 26.
Lastly multiply this answer by 28.
Print the answer and stop.

Now, we note that the factors 20, 22, 24, 26 and 28 are progressively having an increment of 2. Therefore, we may write the algorithm as follows:

Note: In computer language the symbol $*$ denotes multiplication.

$X = 20$

$A = 20$

$X \leftarrow X$ [illegible] [This means, the variable X is assigned a new value, equal to [illegible] value plus 2. Now $X = 22$]

A [illegible] [Now $A = 20 \times 22$]

X [illegible] [Now $X = 22 + 2 = 24$]

$A \leftarrow A * X$ [Now $A = (20 \times 22) \times 24$]

$X \leftarrow X$ [illegible] [Now $X = 24 + 2 = 26$]

$A \leftarrow A * X$ [Now $A = (20 \times 22 \times 24) \times 26$]

$X \leftarrow X + 2$ [Now $X = 26 + 2 = 28$]

$A \leftarrow A * X$ [Now A is the required answer]

Print A and stop.

In this algorithm, we find that the two lines $X \leftarrow X+2$ and $A \leftarrow A * X$ are repeated many times. There is a way to write this briefly, as shown in the flowchart below:

Explanation: The portion

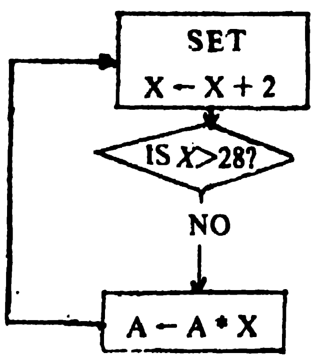


Fig 19.4

is a loop. It is traced again and again. When do we go out of the loop? Only when the answer to the decision box 'Is $X > 28$?' is '*YES*'. At that time the value of A becomes $20 \times 22 \times 24 \times 26 \times 28$.

19.5 Summary

In Example 19.1, we saw an algorithm of the easiest type where the problem involved calculations only. In Example 19.2, we saw algorithm for a problem involving both calculations and decision-making. In Example 19.3, we saw an algorithm that involves calculations, decisions and also loops. In the examples that follow, we take some problems from the earlier chapters of this book and write down algorithms for solving them. The Examples 19.4 to 19.6 will be accompanied by actual working of the algorithm in a particular instance. In the later example, only the flowchart and its method will be given.

Example 19.4

Problem: To find the intersection of a finite number of finite sets.

Method: Let the given sets be $A_1, A_2, \cdots, A_n$.

Arrange them in such a way that A_1 has the least number of elements, when compared with $A_2, \cdots, A_n$ (this facilitates a quicker solution).

Let $A_1 = \{x_1, x_2, \cdots, x_m\}$
Is $x_1 \in A_2$?
If not, leave x_1 and go to x_2.
Else, is $x_1 \in A_3$?
and so on.
If yes for all questions up to "Is $x_1 \in A_n$?", print x_1. Otherwise, do not print x_1.
Then do the same for x_2, x_3 and so on.
The printed elements are the elements of the intersection.

Remark

After Print x_j we have the arrow towards 1
This means, we should go to the step marked 1 namely, the decision box Is $j = m$?
Let us see a particular case of this example.

$$\begin{aligned} \text{Let } A &= \{1,2,3,4,5\} \\ B &= \{3,4,5,8,9\} \\ C &= \{2,3,8,9\} \\ D &= \{1,2,3,10\} \end{aligned}$$

be four sets, for which we want to find the intersection. We rename them as

$$\begin{aligned} A_1 &= C = \{2,3,8,9\} \\ A_2 &= A = \{1,2,3,4,5\} \\ A_3 &= B = \{3,4,5,8,9\} \\ \text{and } A_4 &= D = \{1,2,3,10\} \end{aligned}$$

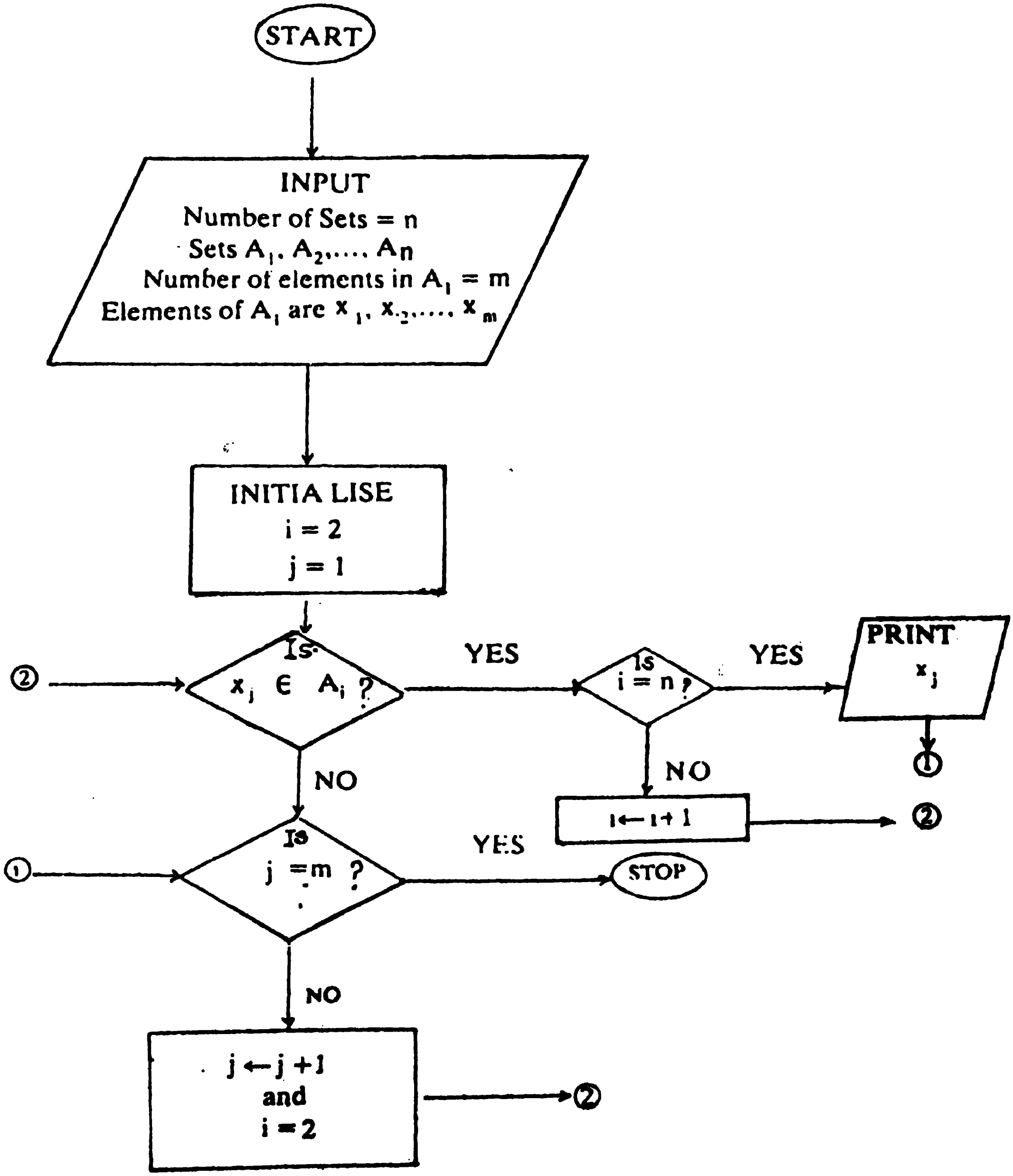

START
INPUT
Number of Sets = n
Sets A1, A2,.... An
Number of elements in A1 = m
Elements of A1 are x1, x2,...., xm
INITIA LISE
i = 2
j = 1
Is xj ∈ Ai ?
YES
Is i = n ?
YES
PRINT xj
1
NO
i ← i + 1
2
2
NO
1
Is j = m ?
YES
STOP
NO
j ← j + 1
and
i = 2
2

Fig. 19.5

Here $m = 4, n = 4, x_1 = 2, x_2 = 3, x_3 = 8, x_4 = 9$
Is $x_1 \in A_2$?
Yes, because $2 \in A_2$.
Is $1 = n$? No, because $n = 4$.
New $i = 3$.
Is $x_1 \in A_3$?
No, because $2 \notin A_3$.
Is $3 = m$? No, because $m = 4$.
New $j = 2$ and $i = 2$.
Is $x_2 \in A_2$? Yes, because $3 \in A_2$.
Is $2 = n$? No, because $n = 4$.
New $i = 3$.
Is $x_2 \in A_3$? Yes $3 \in A_3$.
Is $i = n$? No, $3 \neq 4$.
New $i = 4$.
Is $x_2 \in A_4$? Yes $3 \in A_4$.
Is $i = n$? Yes, $i = n = 4$.
Print x_2. Now 3 is printed.
Is $j = m$? No, $2 \neq 4$.
New $j = 3$ and $i = 2$.
Is $x_3 \in A_2$? No $8 \notin A_2$.
Is $j = m$? No, $3 \neq 4$.
New $j = 4$ and $i = 2$.
Is $x_4 \in A_2$? No, $9 \notin A_2$
Is $j = m$? Yes, both =4.
Stop.
After stopping, what are all the things already printed out? 3 only.
Then we conclude that 3 is the only element of the intersection.
That is, $A \cap B \cap C \cap D = \{3\}$.

Remark

When m and n are large i.e., when too many big sets are given, finding the result by using this algorithm in a computer will be faster than finding the intersection without the computer.

Example 19.5

Problem: Check whether a given finite sequence of numbers is in geometric progression.

Method: Let the given sequence be

$a_1, a_2, \dots, a_n$

This sequence is in G.P. if and only if

$a_i^2 = a_{i-1}a_{i+1}$ for $2 \leq i \leq n-1$.

Step 1: Are there at least three terms? If not, conclude that it is not in G.P. and stop.

Step 2: Else, verify if $a_2^2 = a_1a_3$. If, not, conclude that it is not in G.P. and stop.

Step 3: Else, verify if $a_3^2 = a_2a_4$. If not, conclude that it is not in G.P. and stop. Else, proceed.

Step 4: Go on, until $a_{n-1}^2 = a_{n-2}a_n$ is verified. If yes until that, conclude that it is in G.P. and stop.

Particular Instance: Is 2, 6, 18, 36 in Geometric Progression?

Here, $n = 4$

$a_1 = 2$

$a_2 = 6$

$a_3 = 18$

$a_4 = 36$

Are there at least 3 terms? i.e., is $n > 3$?
Yes,

Initially $i = 2$

$A = 2$

$B = 6$

$C = 18$

Is $B^2 = AC$

$6^2 = 18 \times 2$?

Yes.

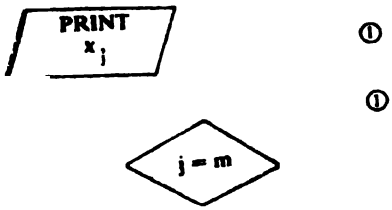
New $i = 3$
Is $i = n$?
No.

Set $A = 6$

$B = 18$

$C = 36$

Is $B^2 = AC$?
i.e., Is $18^2 = 6 \times 36$?
No
PRINT: Not a Geometric Progression.
STOP.


PRINT
x_j
①
①
j = m

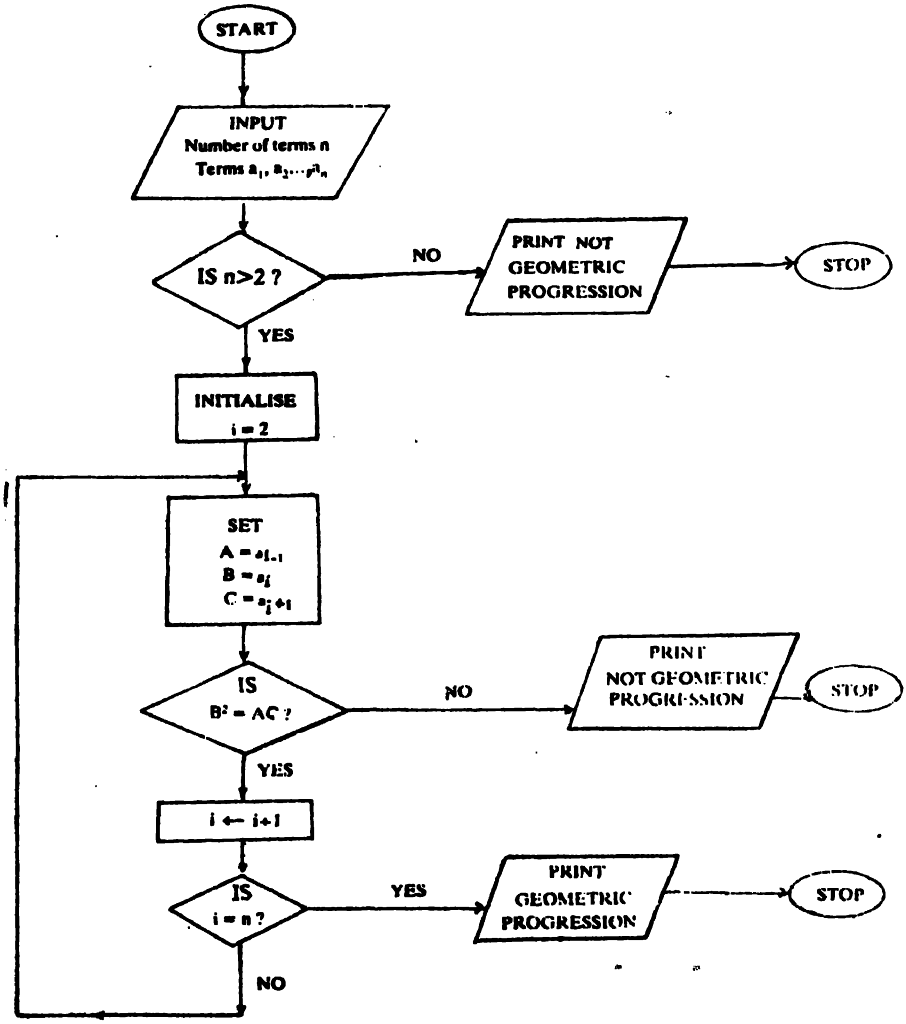

START
INPUT
Number of terms n
Terms $a_1, a_2, \ldots, a_n$
IS n>2 ?
NO
PRINT NOT
GEOMETRIC
PROGRESSION
STOP
YES
INITIALISE
i = 2
SET
$A = a_{i-1}$
$B = a_i$
$C = a_{i+1}$
IS
$B^2 = AC$?
NO
PRINT
NOT GEOMETRIC
PROGRESSION
STOP
YES
i ← i+1
IS
i = n ?
YES
PRINT
GEOMETRIC
PROGRESSION
STOP
NO

Fig. 19.8

Example 19.6

Problem: To check the collinearity of n points.

Method: Let the points be $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$.

The equation of the straight line joining the first two points is

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

This is same as

$$ax + by + c = 0$$

where

$$a = y_1 - y_2$$

$$b = x_2 - x_1$$

and

$$c = x_1(y_2 - y_1) - y_1(x_2 - x_1)$$

Verify if each of the remaining points satisfy this equation.

Particular Example: Verify if the following four points are collinear:

$$(2, 5), (4, 9), (5, 11) \text{ and } (7, 16)$$

Step 1: We enter the inputs $n = 4, x_1 = 2, y_1 = 5, x_2 = 4, y_2 = 9, x_3 = 5, y_3 = 11, x_4 = 7$, and $y_4 = 16$.

Step 2: We compute $a = -4, b = 2, c = -2$

Step 3: Compute $d = ax_3 + by_3 + c$.
$d = 0$ is true

Step 4: $i = 4$
Is $4 > n$? No.

Step 5: Compute $d = ax_4 + by_4 + c = 2 \neq 0$;
hence output: the points are not collinear.

Example 19.7

Problem: Given the value of $\sin\theta$, find absolute value of $\tan\theta$.

Method: Let α be the given value of $\sin\theta$
Then $|\alpha|$ has to be ≤ 1
If $|\alpha| = 1$, that is if $\sin\theta = \pm 1$, then $\cos\theta = 0$ and $\tan\theta$ is not finite.
If $|\alpha| < 1$, then $\tan\theta = \pm\frac{\alpha}{\sqrt{1-\alpha^2}}$ and $|\tan\theta| = \left|\frac{\alpha}{\sqrt{1-\alpha^2}}\right|$
Note also that $|\alpha| > 1$ if and only if $\alpha^2 > 1$.

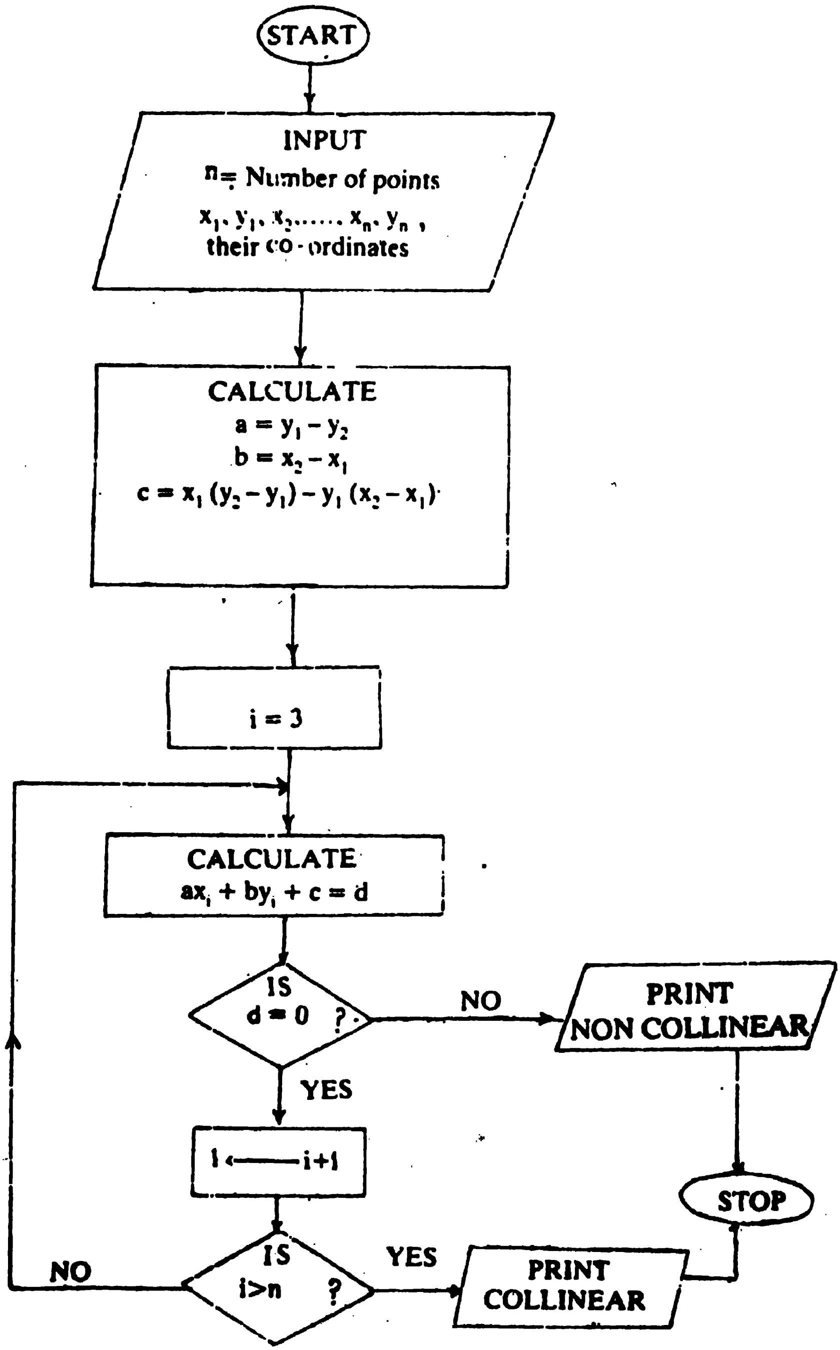

START
INPUT
n= Number of points
x1, y1, x2,...... xn, yn ,
their co-ordinates
CALCULATE
a = y1 - y2
b = x2 - x1
c = x1 (y2 - y1) - y1 (x2 - x1)
i = 3
CALCULATE
axi + byi + c = d
IS
d = 0 ?
NO
PRINT
NON COLLINEAR
YES
i ← i+1
STOP
IS
i>n ?
NO
YES
PRINT
COLLINEAR

Fig. 19.7

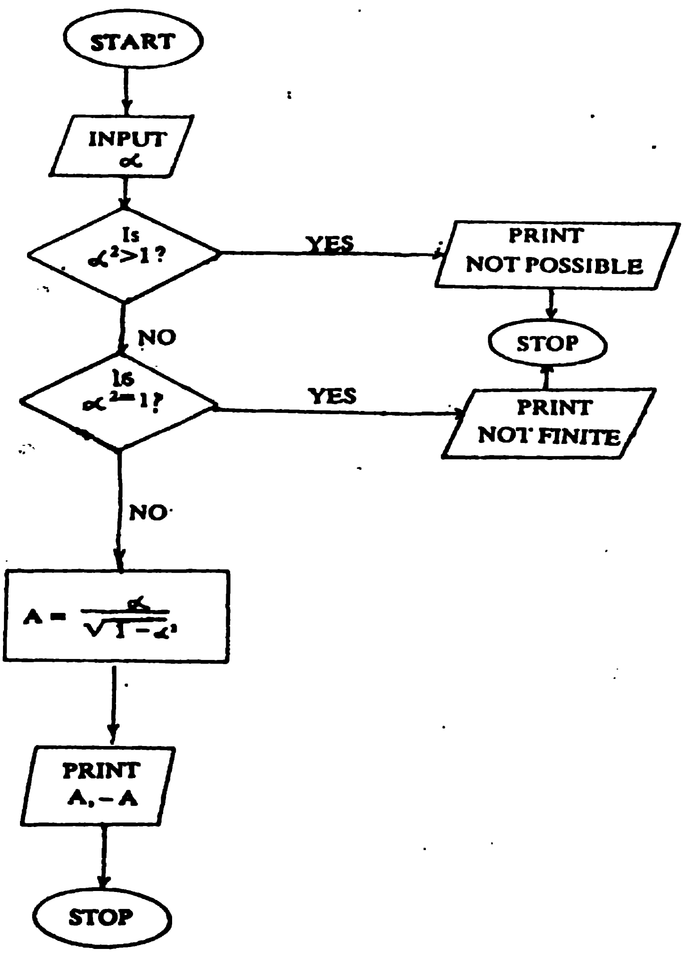

START
INPUT
α
Is
α²>1 ?
YES
PRINT
NOT POSSIBLE
STOP
NO
Is
α²=1?
YES
PRINT
NOT FINITE
NO
A = α/√(1−α²)
PRINT
A, −A
STOP

Fig. 19.8

Example 19.8

Problem: Given two positive integers n and r, compute $C(n,r)$.

Method: $C(n,r) = \dfrac{n(n-1)\cdots(n-r+1)}{1.2\cdots r} = \left(\frac{n}{1}\right)\left(\frac{n-1}{2}\right)\cdots\left(\frac{n-r+1}{r}\right)$

If $r > \frac{n}{2}$, then we use $C(n,r) = C(n,n-r)$.

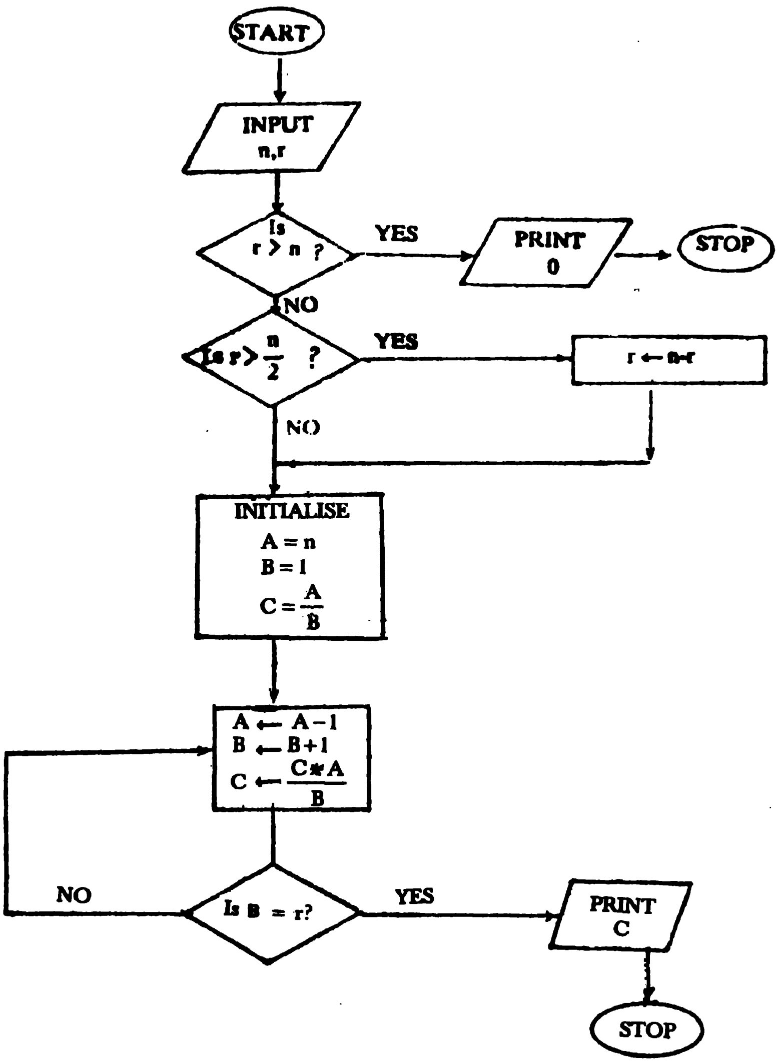


Fig 19.8

Example 19.9

Problem: Calculate the variance and standard deviation for the given raw data.

Method: Variance (VAR) = $\frac{1}{N}\sum(x_i - \bar{x})^2$

N is total number of observations

x_i is ith observation

$\bar{x}$ is arithmetic mean (AM)

$$\text{Standard deviation (SD)} = \sqrt{\frac{\sum(x_i - \bar{x})^2}{N}} = \sqrt{\text{VAR}}$$

Step 1: Take the value of N

Step 2: Take the values of x_i one-by-one and calculate their sum (SX).

Step 3: Calculate the arithmetic mean $AM = \frac{SX}{N}$

Step 4: Calculate the deviations about AM for each observation (say $y_i = x_i - AM$)

Step 5: Calculate the sum of squares ($SS = \sum_{i=1}^{N} y_i^2$)

Step 6: Calculate the variance $VAR = SS/N$ and standard deviation $SD = \sqrt{VAR}$

Observation: The value of S will be zero.

Therefore $\sum(x_i - \bar{x}) = 0$

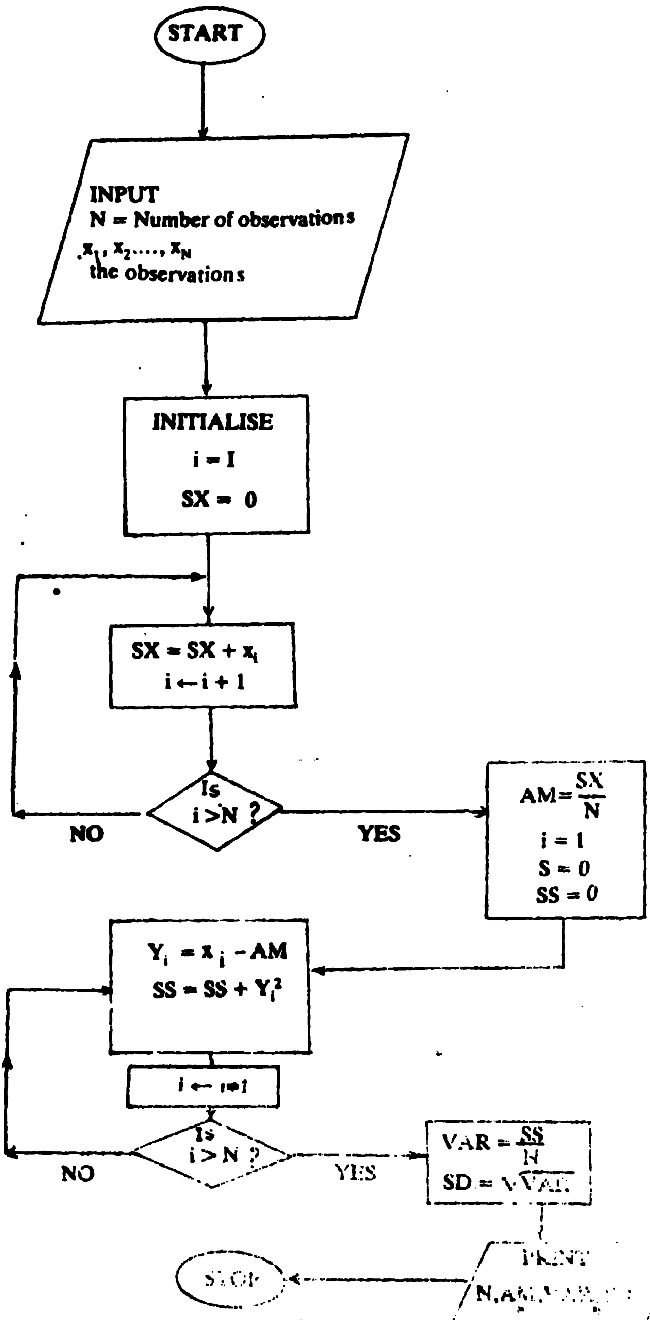

START
INPUT
N = Number of observations
x1, x2,..., xN
the observations
INITIALISE
i = I
SX = 0
SX = SX + xi
i ← i + 1
Is
i > N ?
NO
YES
AM = SX/N
i = 1
S = 0
SS = 0
Yi = xi − AM
SS = SS + Yi2
Is
i > N ?
NO
YES
VAR = SS/N
SD = √VAR

Fig. 19.10

EXERCISE 19.1

Write an algorithm in the form of a flowchart for the following problems:

1. To find the product of the complex numbers $z_1, z_2, \cdots, z_n$ where $n \geq 2$.
2. To find the factorial of a given positive integer.
3. To find if the given sequence $a_1, a_2, \cdots, a_n$ is in arithmetic progression.
4. To calculate the median for raw data $x_1, x_2, \cdots, x_n$ given in the increasing order.
5. To find the angles of a right-angled triangle if the ratio of the two sides that include the right angle is given.
6. To verify whether the four given points $(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)$ lie on a circle.
7. To decide whether the roots of a given quadratic equation are real and distinct.
8. To calculate the arithmetic mean of the given n numbers.

ANSWERS

EXERCISE 13.1

1. $x^6 + 3x^5 + 6x^4 + 7x^3 + 6x^2 + 3x + 1$ **2.** 924 **3.** $58\sqrt{2}$

4. $(-1)C(6,r)x^{12-2r}y^r$

5. $x^{11}+11\frac{x^{10}}{y}+\frac{55x^9}{y^2}+\frac{165x^8}{y^3}+\frac{330x^7}{y^4}+\frac{462x^6}{y^5}+\frac{462x^5}{y^6}+\frac{330x^4}{y^7}+\frac{165x^3}{y^8}+\frac{55x^2}{y^9}+\frac{11x}{y^{10}}+\frac{1}{y^{11}}$

6. 18 **8.** $-35(3x)^4(\frac{x^3}{6})^3$, $35(3x)^3(\frac{x^3}{6})^4$

10. 15 **11.** 55 **12.** $n = 7$; $r = 3$ **13.** 7 and 14

EXERCISE 13.2

1. 995009990004999 **2.** 1·126162419264 **3.** 152 **4.** 96059601

EXERCISE 13.3

1. $\frac{1}{\sqrt{5}}\left[1 - \frac{2x}{5} + \frac{6x^2}{25} - \frac{4x^3}{25} + \dots\right]$

2. $\frac{1}{4^{\frac{1}{3}}}\left[1 + \frac{x^2}{4} + \frac{x^4}{8} + \frac{7x^6}{96} + \dots\right]$, valid when $-\frac{2}{\sqrt{3}} < x < \frac{2}{\sqrt{3}}$.

3. $\frac{15015}{16}$ **7.** $a = 1$; $b = -\frac{35}{24}$ **8.** .996672

9. 1.06 **11.** $a = 2$; $b = 12$ **12.** -2 and 3

Miscellaneous Exercise on Chapter 13

2. 12 **4.** $\frac{1}{2}$; $\frac{x^3}{16}$ **6.** $\frac{1}{9}(2 + \frac{11x}{3} + 4x^2 + \dots)$ **7.** 16

EXERCISE 14.1

1. 2.7 2. $\frac{b^n}{n!}e^a$ 3. $\frac{1}{2}(e+\frac{1}{e})$ 4. $e-1$

6. (i) $e-2$ (ii) $e-\frac{5}{2}$ (iii) $\frac{1}{2}\left(e-\frac{1}{e}\right)$ (iv) $\frac{1}{2}\left(e-\frac{1}{e}\right)-1$

7. (i) 2e (ii) $\frac{3e}{2}$ (iii) $\frac{1}{e}$

8. (i) $\frac{e}{2}-1$ (ii) e^2-1

9. (i) $x+\frac{x^3}{3!}+\frac{x^5}{5!}+\dots$ (ii) $2\left[1+\frac{(2x)^2}{2!}+\frac{(2x)^4}{4!}+\dots\right]$

10. 2e

EXERCISE 14.2

7. 1.3

Miscellaneous Exercise on Chapter 14

1. e^x-e^y 2. 0.7788

EXERCISE 15.1

1. (i) $\frac{3}{5}, \frac{4}{5}, 1$
   (ii) $\frac{3}{4}, \frac{4}{3}$, not defined
   (iii) 216 sq. units
   (iv) $\frac{1}{3}, \frac{1}{2}, 1$

EXERCISE 15.3

6. (i) $r=6, A=36°52', B=53°8', C=90°$
   (ii) $r=1.5, A=53°8', B=14°14', C=112°38'$

8. 161.16 cm^2 (nearly), 307.92 cm^2 (nearly) and 1018.81 cm^2 (nearly)

EXERCISE 15.4

1. $B=37°, b=32$ (nearly), $c=54$ (nearly)

2. $B=64°, b=5.64, a=2.85$ (nearly)

3. $A=62°32', B=27°28', c=56.8$

4. $A=41°$, (nearly), $B=49°$ (nearly), $a=2.9$ (nearly)

5. $B=51°, a=3.6$ (nearly), $c=5.8$ (nearly)

6. $A=42°29', B=47°31', b=450$

EXERCISE 15.5

1. $58°59'33''$ (nearly) 2. $A = 45°, B = 120°, C = 15°$

3. $104°28'39''$ (nearly) 4. $A = 56°15'4'', B = 59°51'10'', C = 63°53'46''$.

EXERCISE 15.6

1. $A = 117°38'45''$, $B = 27°38'45''$

2. $B = 71°44'30''$, $C = 48°15'30''$

3. $c = \sqrt{6}$, $A = 105°$, $B = 15°$

4. $b = 40$, $A = 30°$, $C = 120°$

EXERCISE 15.7

1. There is no triangle

2. $B_1 = 15°, C_1 = 135°, b_1 = 50(\sqrt{6} - \sqrt{2})$
$B_2 = 105°, C_2 = 45°, b_2 = 50(\sqrt{6} + \sqrt{2})$

3. $c = 4\sqrt{3} \pm 2\sqrt{5}$

EXERCISE 15.8

1. (i) $a = 70$ (nearly), $b = 76$ (nearly), $C = 59°$
(ii) $b = 40.5$, $c = 31.15$, $C = 47°$
(iii) $a = 274$ (nearly), $C = 339$ (nearly), $B = 57°50'$

EXERCISE 15.9

1. 18.12 m 2. 17.32 m; 10m 3. 260.51m 4. 225 m 5. 227.39 m
6. 68.3 m 7. S $54°9'43''N$; 22.2 km 8. 146.4 m 9. 17.32 m
10. .36.54 m, 1015.37 m 11. 1366 m

EXERCISE 16.1

1. (i) $\frac{-\pi}{2}$ (ii) $\frac{-\pi}{6}$ (iii) $\frac{2\pi}{3}$ (iv) $\frac{\pi}{6}$ (v) $\frac{\pi}{6}$ (vi) $\frac{-\pi}{6}$

11. $\frac{x+y}{1-xy}$, where $xy < 1$ 12. (i) $\pm\frac{1}{\sqrt{2}}$ (ii) $\frac{\pi}{4}$
13. (i) $\sin^{-1}\frac{x}{a}$ (ii) $\frac{x}{2}$ (iii) $\frac{\pi}{4} - x$ (iv) $3\tan^{-1}\frac{x}{a}$

Miscellaneous Exercise on Chapters 15 and 16

5. $(2n\pi - \frac{\pi}{3})$, $(2n\pi \pm \frac{\pi}{2})$ 6. $\frac{24}{25}$ 7. $\sqrt{\frac{3}{28}}$

22. $2n\pi + \frac{\pi}{4} \pm A$

EXERCISE 17.1

2. (a) 12.75

4. 38.9 paise, Rs 0.39 (See Problem 5 for the explanation)

6. (a) 8.8 years, 8.2 years (b) The concentration of data in Class IV of school A is much higher than that of Class IV of school B.

7. The mean age is 23.6 years approximately.

8. 40.6 mm (approx.)

EXERCISE 17.2

1. Median = 74th observation = 12. Thus, for about half the number of days, more than 12 students were absent.

2. Median = $\frac{23+24}{2}$ = 23.5. About 50% of the students obtained less than 23.5 marks out of 50 in the test.

3. First note that the class 15-19 represents "15 years or more but less than 20". Median = 24.5 years. Nearly half the women were married between the ages 15 and 24.5.

4. The distribution of data is not symmetrical.

EXERCISE 17.3

2. (a) and (b) The variance is the same in both the cases and equal to the variance of the original scores obtained.

4. (a) 69.16
   (b) Variance in case (b) is 4 times the variance in case (a). (See Problem 5 of this exercise for the reason.)

6. $\sigma^2 = 140.70, \bar{x} - \sigma = 78.58, \bar{x} + \sigma = 102.30$, required per cent = 64.6 (You may subtract 92 from each score to make calculations easier.)

EXERCISE 17.4

1. $M = \frac{49.4 + 49.6}{2} = 49.5\text{cm}$, $d = 21.5\text{cm}$, 27.

2. $M = 67.1\text{m}^2$, $d = 20.9\text{m}^2$, 48.4%, 97%

EXERCISE 18.1

1. Polygon with vertices (0,0), (3,0), $\left(\frac{12}{5}, \frac{2}{5}\right)$, $(0, 1)$

2. Polygon with vertices $(1,0), (6,0), (6, \frac{7}{3}), (\frac{18}{7}, 5), (0,5), (0,1)$

5. $x \geq 0, 2x + y \geq 2, x - y \leq 1, x + 2y \leq 8, y \geq 0$

6. $x \geq 0, 2x + 3y \geq 3, y \geq 0, x - 6y \leq 3, 3x + 4y \leq 18, -7x + 4y \geq 14$

EXERCISE 18.2

1. The set of feasible solutions is a polygon with vertices $\left(\frac{18}{7}, \frac{2}{7}\right), \left(\frac{7}{2}, \frac{3}{4}\right), \left(\frac{3}{2}, \frac{15}{4}\right)$, $\left(\frac{3}{13}, \frac{24}{13}\right)$. Hence, max. = 19, min. = $\frac{63}{13}$.

2. Max. = $14\frac{1}{13}$ is at $\left(\frac{84}{13}, \frac{15}{13}\right)$, and Min. = $3\frac{1}{14}$ is at $\left(\frac{9}{14}, \frac{25}{14}\right)$. Even though the set of feasible solutions is not a closed polygon the minimum can still be found by the graphical method. Min. $9\frac{2}{3}$ is at $\left(\frac{8}{3}, \frac{1}{3}\right)$.

3. Let x km = distance travelled at 25 km/h
   y km = distance travelled at 40 km/h

   We maximise $x + y$ subject to the constraints

$$\begin{aligned} 2x + 5y &\leq 100, \\ \frac{x}{25} + \frac{y}{49} &\leq 1, \\ x, y &\geq 0. \end{aligned}$$

   Max. = 30 km with $x = \frac{50}{3}, y = \frac{40}{3}$

4. If x, y denote quantities (in grams) of wheat and rice, our problem is to minimise $\frac{2x}{1000} + \frac{8y}{1000}$ subject to the constraints

$$\begin{aligned} 0.1x + 0.05y &\geq 50 \\ 0.25x + 0.5y &\geq 200 \\ x, y &\geq 0 \end{aligned}$$

   Diet cost is minimum (Rs 2.40) when $x = 400$g, $y = 200$g.